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On the Capacity of Feed-forward Neural Networks for Fuzzy Classification

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Abstract-This paper investigates the ability of feed-forward neural network (FFNN) classifiers trained with examples to generalize and estimate the structure of the feature space in the form of class membership information. A functional theory of FFNN classifiers is developed from formal definitions. The properties of discriminant functions learned by FFNN classifiers from sample data are also studied. These properties show that the ability of FFNNs to identify and quantify uncertainty in a feature space is sensitively dependent on the topology of the feature space and that FFNNs trained to classify overlapping classes of data tend to create sharp transitions between closely spaced feature vectors belonging to different classes.

Key Words: discriminant function; feed-forward neural network; fuzzy classification; learning; membership profile; uncertainty.

1 Introduction

Feed-forward neural networks (FFNNs) have been a natural choice as trainable pattern classifiers because of their function approximation capability and generalization ability [2, 5, 6]. Model-free estimation of the functional relationship between the input and the output, together with other desirable properties such as generalization ability, robustness to outliers and ease of design have motivated several applications for FFNN classifiers [1, 13, 17].

Feed-forward neural networks are often trained to function as classifiers by minimizing an objective function through gradient descent. In general, any gradient-descent-based algorithm is prone to behave erratically when used to train FFNNs to classify overlapping classes of feature vectors [12]. In order to facilitate the network to converge to a solution when the training set contains overlapping classes, Keller and Hunt [12] incorporated fuzzy principles into the training algorithms for FFNN classifiers. The input to this network was formed by the feature vectors of the training data set. The 'desired output' vector for training the network was computed from the membership functions chosen heuristically for each pattern class. These membership functions give a measure of the extent to which a certain pattern belongs to each class. Thus each pattern may belong to many of a specified number of classes and the network was trained to give these membership values as the outputs.

Pal and Mitra [16] trained a multilayered FFNN to function as a fuzzy classifier for a data set consisting of overlapping classes of data. The input vector to the network was formed by mapping the actual n-dimensional feature vector to a $3n$-dimensional space, with each component of the actual feature vector being mapped to three fuzzy sets representing

high, *medium* and *low* values in its domain. This allowed the network to handle uncertain and ambiguous information as well as to take linguistic inputs. The 'desired output' for training the network was computed from membership functions chosen *a priori* for each pattern class. The grade of membership values given by the network, together with any incidental contextual information, can be used to make a hard decision later.

In both training schemes described above, the membership functions have to be estimated or chosen *a priori.* More significantly, a classifier decision function for a given feature space needs to be determined over the entire feature space, or at least over a dense subset of the feature space. The given data set is a finite subset of this dense subset of the feature space. Suppose a conventional FFNN is trained on this finite subset of data points. It is expected that the network will learn the value of the *unknown classifier decision function* at least for all the points of a dense subset of the feature space. It is in this sense that the FFNN is expected to 'generalize' [2, 5, 6]. Suppose the class membership functions are known and represent a measure of the extent to which a certain feature vector belongs to each class. Then the input-output function is *exactly* known over the entire feature space. This information is available in a functional form, with all the parameters already estimated. Therefore, in this scheme the network is capable of handling uncertain (linguistic) inputs, but does not necessarily perform an estimation (generalization) task. However, model-free estimation of the input-output relationship is the main import of the neural processing paradigm [13].

Takagi and Hayashi [23] used FFNNs *per se* as estimators of class membership functions. In their approach, the training data were pre-clustered and labeled for supervised training. The number of input nodes of the FFNN was set equal to the dimension of the feature space and the number of output units was set equal to the number of classes in the pre-clustered data. Each feature vector in the training set belonging to the ith class was given as the input and the network was trained to respond with a value of 1 at the ith output unit and a value of 0 at all the other output units. It was assumed that when a feature vector not contained in the training set is input to the network, the network will respond with output values between 0 and 1 for each class because of its generalization ability. It was also assumed that these output values were consistent with the location of this feature vector relative to the training data. This membership estimation procedure was used in fuzzy inferencing algorithms for practical applications. However, this approach may not be valid when the classes of data are closely spaced or overlapping, as is often the case in practical situations [3]. In general, the ability of FFNNs trained by examples to function as fuzzy classifiers by themselves is sensitively dependent on the following: 1) their size, 2) the training data set, and 3) the learning algorithm.

This paper investigates the ability of FFNNs trained with sample data to function as estimators of class membership profiles. In Section 2, some formal definitions are introduced and a mathematical model of the training and learning aspects of FFNN classifiers is developed. Section 3 analyzes the properties of the discriminant functions learned by FFNN classifiers in relation to the topology of the feature space. Section 4 contains concluding remarks.

2 Feed-forward Neural Network (FFNN) Classifiers

Consider a FFNN with n_i inputs, n_o output units and one layer of n_h hidden units. Let $\mathbf{v}_j^T = [v_{j1}, v_{j2}, \ldots, v_{jn_i}]$ be the weight vector connecting the jth hidden unit to the inputs and $\mathbf{w}_i^T = [w_{i1}, w_{i2}, \ldots, w_{in_h}]$ be the weight vector connecting the ith output unit to the hidden units. Let $\mathbf{V}$ be the matrix with the vectors $\mathbf{v}_j$ as its columns, and $\mathbf{W}$ the matrix with the vectors $\mathbf{w}_i$ as its columns. Let the activation function of the hidden units be the sigmoidal function $g : \mathbb{R} \to [0, 1]$.

Definition 1: *A conventional feed-forward neural network* is defined as the function $\mathcal{N} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_o}$ which maps $\mathbf{x} = [x_1, x_2, \ldots, x_{n_i}]^T$ to $\mathcal{N}(\mathbf{V}, g, \mathbf{W};\, \mathbf{x})$, such that

$$\Pi_i \mathcal{N}(\mathbf{V}, g, \mathbf{W};\, \mathbf{x}) = \sum_{j=1}^{n_h} w_{ij}\, g\left(\sum_{\ell=1}^{n_i} v_{j\ell}\, x_\ell + v_{j0}\right) + w_{i0} \;\; \forall i, \tag{1}$$

where $v_{j0}, w_{i0} \in \mathbb{R}\ \forall\, j, i$, and Π_i is the ith co-ordinate function $\Pi_i : \mathbb{R}^{n_o} \to \mathbb{R}$.

In Definition 1, $g(\mathbf{v}_j^T \mathbf{x} + v_{j0}) = g(\sum_{\ell=1}^{n_i} v_{j\ell}\, x_\ell + v_{j0}) = \hat{h}_j$ is called the *response of the jth hidden unit* for the input $\mathbf{x}$. Similarly, $\sum_{j=1}^{n_h} w_{ij} \hat{h}_j + w_{i0} = \bar{y}_i$ is called the *response of the ith output unit* for the input $\mathbf{x}$. As a matter of convention, $\mathcal{N}(\mathbf{V}, g, \mathbf{W};\, \mathbf{x})$ will be henceforth denoted as $\mathcal{N}(\mathbf{x})$. The function approximation capability of conventional FFNNs has been investigated by many authors [2, 5, 6] and can be stated in general terms as follows [13, 24]:

Theorem 1: Suppose $F : \mathbb{R}^{n_i} \to \mathbb{R}^{n_o}$ is any Borel-measurable function defined on a compact subset $\mathcal{K}$ of $\mathbb{R}^{n_i}$ and $\{\, (\mathbf{x}_k, F(\mathbf{x}_k))\, \}_{k=1}^m$ is a set of m samples of this function. Given an $\varepsilon > 0$, there exist a conventional FFNN $\mathcal{N}(\cdot)$ with n_h hidden units and an integer n_h^0 such that

$$\sum_{k=1}^{m} d_{n_o}\left(\mathcal{N}(\mathbf{x}_k), F(\mathbf{x}_k)\right) < \varepsilon, \quad \forall n_h > n_h^0, \tag{2}$$

and for any metric d_{n_o} on $\mathbb{R}^{n_o}$. The quantity ε includes the representation, generalization and optimization errors in the realization of $\mathcal{N}$ [13, 24].

Theorem 1 proves that FFNNs with a single layer of nonlinear hidden units are adequate for most practical applications and, as such, research efforts often focus on FFNNs with a single hidden layer [2, 5, 6, 13]. Given a sufficient number of samples of a Borel-measurable function F defined on a compact set $\mathcal{K}$, the process of finding a $\mathcal{N}(\cdot)$ which approximates F to a desired amount of accuracy is called *training a conventional FFNN*. Therefore, the process of *training* a conventional FFNN to *learn* a given function is essentially the search for the appropriate number of hidden units n_h and the corresponding weight matrices $\mathbf{V}$ and $\mathbf{W}$.

Definition 2: Suppose $F : \mathbb{R}^{n_i} \to \mathbb{R}^{n_o}$ is any Borel-measurable function defined on a compact subset $\mathcal{K}$ of $\mathbb{R}^{n_i}$ and $\{\, (\mathbf{x}_k, F(\mathbf{x}_k))\, \}_{k=1}^m$ is a set of m samples of this function. A

conventional FFNN is said to have *learned* F from the given samples to within an error $\varepsilon > 0$ if it has n_h hidden units and weight matrices $\mathbf{V}$ and $\mathbf{W}$ such that

$$\sum_{k=1}^{m} d_{n_o}\left(\mathcal{N}(\mathbf{x}_k), F(\mathbf{x}_k)\right) < \varepsilon. \tag{3}$$

According to Definition 2, any FFNN which is a "close" approximator of a given function can be said to have learned the function.

It is noted here that for most nontrivial Borel-measurable functions, the amount of approximation accuracy achievable with a fixed number of hidden units cannot be predetermined. In addition, for many functions the number of hidden units required may be physically unrealizable [2, 5, 6, 13]. If the number of hidden units is fixed, the desired amount of approximation accuracy may not be achievable and the training process may not terminate. On the other hand, carrying out a simultaneous search for the appropriate number of hidden units and the weight matrices $\mathbf{V}$ and $\mathbf{W}$ is a difficult task for most functions of practical significance. In practice, most training algorithms for FFNNs chose the metric d_{n_o} in equation (2) as the Euclidean norm and train the network till ε is some very small value. These training algorithms necessarily adapt the synaptic weights [22, 24], and in some cases they also adapt the number of hidden units [8, 24]. Therefore, the *termination criterion* for the training process can be stated in general terms as follows:

Train the network till the condition,

$$\sum_{k=1}^{m} d_{n_o}\left(\mathcal{N}(\mathbf{x}_k), F(\mathbf{x}_k)\right) < \varepsilon, \tag{4}$$

is satisfied.

Consider a data set of m feature vectors $\mathbf{x} \in \mathcal{X}$, where $\mathcal{X}$ is an N-dimensional feature space. Suppose $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n \subset \mathcal{X}$ are n known classes of feature vectors in the feature space $\mathcal{X}$. Let $\mathcal{X}$ be a compact metric subspace of $\mathbb{R}^N$ and $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ be Borel sets of $\mathbb{R}^N$. Therefore, $\mathcal{X}$ is a Borel set of $\mathbb{R}^N$ and so are $\mathcal{X} - \mathcal{C}_i$ for $i = 1, 2, \ldots, n$. This data set of feature vectors may be simply referred to as the set $\mathcal{X}$.

Definition 3: The *classifier* for the data set $\mathcal{X}$ is defined as the function $F : \mathcal{X} \to \{0, 1\}^n$ such that for $i = 1, 2, \ldots, n$,

$$\Pi_i F(\mathbf{x}) = \begin{cases} 1 \text{ if } \mathbf{x} \in \mathcal{C}_i \\ 0 \text{ otherwise.} \end{cases} \tag{5}$$

Definition 3 takes into account the case of overlapping classes of data sets also. In particular, suppose the classes $\mathcal{C}_p$ and $\mathcal{C}_q$ are overlapping, i.e., $\mathcal{C}_p \cap \mathcal{C}_q \neq \oslash$. In this case, Definition 3 implies that $\forall\, \mathbf{x} \in \mathcal{C}_p \cap \mathcal{C}_q$, $\Pi_p F(\mathbf{x}) = \Pi_q F(\mathbf{x}) = 1$.

Conventionally, FFNNs are trained to function as pattern classifiers by giving feature vectors belonging to the ith class as the input and adjusting the weights to obtain a response

of '1' from the ith output unit and a response of '0' from the other output units. This procedure is repeated for all the feature vectors belonging to all the classes in one *training epoch* [13, 16, 22].

Proposition 1: A conventional FFNN classifier is capable of *learning to classify* the data set $\mathcal{X}$, i.e., learning the classifier function F (in Definition 3).

Proof: Since $\{\mathbf{x} : \Pi_i F(\mathbf{x}) \geq a\}$ is either $\oslash$ or $\mathcal{C}_i$ or $\mathcal{X}$ for any $a \in \mathbb{R}$, F is a Borel-measurable function. Further, $\mathcal{X}$ is defined to be a compact subset of $\mathbb{R}^N$.

Definition 4: The *conventional FFNN classifier* for the data set $\mathcal{X}$ is defined as the conventional FFNN that has learned the classifier function of the data set $\mathcal{X}$.

Clearly, for a conventional FFNN classifier, the number of inputs n_i equals the dimension N of the feature space and the number of output units n_o equals the number of classes n in the data set $\mathcal{X}$. In the following, a *conventional FFNN classifier* may be simply referred to as a conventional FFNN or FFNN.

As mentioned already, when a conventional FFNN is to be trained to function as a classifier, *hard training* involving a simultaneous search for the appropriate number of hidden units and the weight matrices is often not necessary [16, 23, 24]. In this case, the network can be *soft trained* to learn to *consistently partition the feature space* as explained in the definition below.

Definition 5: A conventional FFNN is said to have learned to *consistently partition the feature space* of a given data set if for $i = 1, 2, \ldots, n$, $\Pi_i \mathcal{N}(\mathbf{x}) \in [0, 1]$ and

$$\Pi_i \mathcal{N}(\mathbf{x}) \begin{cases} > \ \gamma & \text{if} \quad \mathbf{x} \in \mathcal{C}_i \\ \leq \ \gamma & \text{otherwise,} \end{cases} \tag{6}$$

where $\gamma \in$ (0,1). Usually, for symmetry and simplicity, the value of γ may be taken to be 1/2.

One way to ensure that the first condition in Definition 5 is satisfied, i.e., for $i = 1, 2, \ldots, n$, $\Pi_i \mathcal{N}(\mathbf{x}) \in [0, 1]$, is to introduce a suitable nonlinearity in the output units of $\mathcal{N}$. Accordingly, the *bounded FFNN* $\hat{\mathcal{N}}$ is defined as the function $\hat{\mathcal{N}} : \mathbb{R}^{n_i} \rightarrow [0, 1]^{n_o}$ which maps $\mathbf{x} = [x_1, x_2, \ldots, x_{n_i}]^T$ to $\hat{\mathcal{N}}(\mathbf{V}, g, \mathbf{W}; \mathbf{x})$, such that

$$\Pi_i \hat{\mathcal{N}}(\mathbf{V}, g, \mathbf{W}; \mathbf{x}) = \sigma \left(\sum_{j=1}^{n_h} w_{ij} g \left(\sum_{\ell=1}^{n_i} v_{j\ell}\, x_\ell + v_{j0} \right) + w_{i0} \right) \ \forall i, \tag{7}$$

where $\sigma : \mathbb{R} \rightarrow [0, 1]$ is a sigmoidal function. As a convention, $\hat{\mathcal{N}}(\mathbf{V}, g, \mathbf{W}; \mathbf{x})$ will be henceforth denoted as $\hat{\mathcal{N}}(\mathbf{x})$. The following proposition proves that the function approximation capability of $\mathcal{N}$ is preserved in $\hat{\mathcal{N}}$, which implies that the nonlinearity σ of the output units

merely limits the range of $\mathcal{N}$ and $\hat{\mathcal{N}}$ can approximate any Borel-measurable function with range $[0,1]$.

Proposition 2: Suppose $F : \mathbb{R}^{n_i} \rightarrow [0,1]^{n_o}$ is any Borel-measurable function defined on a compact subset $\mathcal{K}$ of $\mathbb{R}^{n_i}$ and $\{ \, (\, \mathbf{x}_k, F(\mathbf{x}_k) \,) \, \}_{k=1}^{m}$ is a set of m samples of this function. Given an $\varepsilon > 0$, there exist a bounded FFNN $\hat{\mathcal{N}}$ with n_h hidden units and an integer n_h^0 such that

$$\sum_{k=1}^{m} d_{n_o} \left(\hat{\mathcal{N}}(\mathbf{x}_k), F(\mathbf{x}_k) \right) < \varepsilon, \quad \forall n_h > n_h^0, \tag{8}$$

and for any metric d_{n_o} on $\mathbb{R}^{n_o}$. The quantity ε includes the representation, generalization and optimization errors in the realization of $\hat{\mathcal{N}}$.

Proof: Since $\sigma : \mathbb{R} \rightarrow [0,1]$ is a sigmoidal function, it is 1-to-1, continuous, and increasing. Therefore, $\sigma^{-1} : [0,1] \rightarrow \mathbb{R}$ is also 1-to-1, continuous, and increasing. Define $G : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_o}$ as $\Pi_i G = \sigma^{-1} \left(\Pi_i F \right)$ for $i = 1, 2, \ldots, n_o$. G is Borel-measurable and is defined on the same compact set as F. Since σ is continuous, given an $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sum_{k=1}^{m} d_{n_o} \left(\mathcal{N}(\mathbf{x}_k), G(\mathbf{x}_k) \right) < \delta \longrightarrow \sum_{k=1}^{m} d_{n_o} \left(\hat{\mathcal{N}}(\mathbf{x}_k), F(\mathbf{x}_k) \right) < \varepsilon. \tag{9}$$

By Theorem 1, there exist matrices $\mathbf{V}$ and $\mathbf{W}$ and an integer n_h^o such that

$$\sum_{k=1}^{m} d_{n_o} \left(\mathcal{N}(\mathbf{x}_k), G(\mathbf{x}_k) \right) < \delta, \;\; \forall n_h > n_h^0. \tag{10}$$

Clearly, training a conventional FFNN to consistently partition the feature space of a given data set is a much easier task than training the network to learn the classifier function. However, the difficulty with such training is that there is no explicit function F that can be used in the termination criterion (2) to check if the network has learned to consistently partition the feature space to the desired amount. The following criterion does not explicitly involve the function to be learned and is therefore suitable for training conventional FFNNs to learn to consistently partition a feature space.

Train the network till the condition,

$$\mathcal{L}_1(\mathcal{N}, \mathcal{X}) = \frac{1}{mn} \sum_{i=1}^{n} \left[\frac{1}{(1-\gamma)} \sum_{\forall \mathbf{x} \in \mathcal{C}_i} (\, \Pi_i \mathcal{N}(\mathbf{x}) - \gamma \,) + \frac{1}{\gamma} \sum_{\forall \mathbf{x} \, \notin \, \mathcal{C}_i} (\, \gamma - \Pi_i \mathcal{N}(\mathbf{x}) \,) \right] \geq \eta, \tag{11}$$

is satisfied for some $\eta \in [0,1]$.

It is obvious that if the network has learned to consistently partition the feature space, then it satisfies inequality (11). This termination criterion is based on a global measure of how well the network has learned to function as a classifier. It quantifies the goodness-of-fit that the

network has achieved in extrapolating the classifier function, and relates it to the 'hardness' of training that the network has been subjected to. The quantity $\mathcal{L}_1(\mathcal{N}, \mathcal{X})$ measures the "goodness-of-fit" the network achieves for the function it is trained to learn. The parameter η is the *hardness* of training that the network is subjected to in order to achieve this goodness-of-fit. The parameter η comprehensively models the training and learning aspects of the FFNN classifier. In the ideal situation where the optimal network size can be estimated and an efficient training algorithm can be used, the designer sets the value of η to 1. In this case, the training process is guaranteed to terminate in finite time and the FFNN learns to accurately assign each feature vector to a certain class. On the other hand, if the designer sets the value of η close to 0 in order to accommodate for the suboptimality of the network size and the learning algorithm, then the network is only minimally trained, or not trained at all. In other words, the parameter η summarizes the effects of the suboptimality of the network size and the learning algorithm on the ability of the network to perform the classification task it is trained to perform. Therefore, this criterion measures the inherent ability of the network to function as the classifier by balancing the hardness of training required against the degree of approximation achieved. The parameter η can also be interpreted as a measure of the network's robustness to noise. The network has maximum robustness to noise if it is trained to satisfy (11) with $\eta = 1$. In this case, apart from learning to consistently partition the feature space, the network also learns the classifier function in Definition 3 and can therefore function as the classifier.

Proposition 3: Suppose a conventional FFNN has learned to consistently partition the feature space of the data set $\mathcal{X}$. Then it satisfies the termination criterion (11) with $\eta = 1$ if and only if it has learned the classifier function F.

Proof: The proof of Proposition 3 is presented in Appendix A.

The termination criterion (11) is valid for very general data sets, including those consisting of nonconvex and overlapping classes. However, other criteria that are computationally less expensive can be established for particular types of data sets. The following criterion requires fewer computations than (11) and is particularly useful when the data set consists of nonoverlapping or minimally overlapping classes.

Train the network till the condition,

$$\mathcal{L}_2(\mathcal{N}, \mathcal{X}) = \frac{1}{m(n-1)} \sum_{i=1}^{n} \sum_{\forall \mathbf{x} \in \mathcal{C}_i} \sum_{\forall j \neq i} (\ \Pi_i \mathcal{N}(\mathbf{x}) - \Pi_j \mathcal{N}(\mathbf{x})\ ) \geq \eta, \tag{12}$$

is satisfied for some $\eta \in [0, 1]$. The parameter η is a measure of the network's robustness to noise. The network has maximum robustness to noise if $\eta = 1$.

Proposition 4: Suppose a conventional FFNN has learned to consistently partition the feature space of the data set $\mathcal{X}$ consisting of nonoverlapping classes of data. Then it satisfies the termination criterion (12) with $\eta = 1$ if and only if it has learned the classifier function F.

Proof: The proof of Proposition 4 is presented in Appendix B.

Alternative criteria for terminating the soft training of the network can be stated as

$$\frac{1}{m(1-\gamma)} \sum_{i=1}^{n} \sum_{\forall \mathbf{x} \in \mathcal{C}_i} (\ \Pi_i \mathcal{N}(\mathbf{x}) - \gamma \ ) \geq \eta_d, \tag{13}$$

where η_d is the *power of detection* and as

$$\frac{1}{m(n-1)^2(1-\gamma)} \sum_{i=1}^{n} \sum_{\forall j \neq i} \sum_{\forall \mathbf{x} \in \mathcal{C}_j} (\ \gamma - \Pi_j \mathcal{N}(\mathbf{x}) \ ) \geq \eta_r, \tag{14}$$

where η_r is the *power of rejection.* Again, the closer η_d or η_r are to 1, the more robust the network is. Another computationally simple criterion can be stated as

$$\sum_{i=1}^{n} \sum_{\mathbf{x} \in \mathcal{X}} (\ \Pi_i \mathcal{N}(\mathbf{x}) - \gamma \ )^2 \geq M. \tag{15}$$

Propositions 3 and 4 bring out the relationship between a network being *soft-trained* to learn to consistently partition the feature space and it being *hard-trained* to learn the classifier function F. Apart from the obvious practical advantages such as fast training, smaller size, etc., soft-training a network also results in other radical qualitative and quantitative differences in so far as the ability of the network to identify and quantify the uncertainty inherent in the data set is concerned. These differences are investigated in the rest of this section.

Lemma 1: Suppose a conventional FFNN has learned to consistently partition the feature space of $\mathcal{X}$. Then there exists an $\alpha \in [0,1]$ such that for $i = 1, 2, \ldots, n$,

$$\Pi_i \mathcal{N}(\mathbf{x}) \begin{cases} > & \gamma + \alpha(1-\gamma) & \text{if } \mathbf{x} \in \mathcal{C}_i \\ \leq & \gamma - \alpha\gamma & \text{otherwise.} \end{cases} \tag{16}$$

In this case, the training process will terminate if equation (11) is used as the termination criterion with $\eta \leq \alpha$.

Proof: The proof of Lemma 1 is presented in Appendix C.

Lemma 1 gives the bounds on the response of the FFNN trained to consistently partition the feature space of a data set.

Suppose a conventional FFNN satisfies the condition that for $i = 1, 2, \ldots, n$,

$$\Pi_i \mathcal{N}(\mathbf{x}) \begin{cases} > & \gamma + \eta(1-\gamma) & \text{if } \mathbf{x} \in \mathcal{C}_i \\ \leq & \gamma - \eta\gamma & \text{otherwise,} \end{cases} \tag{17}$$

for some $\eta \in [0,1]$. Then, since $\gamma + \eta(1-\gamma) \geq \gamma$ and $\gamma - \gamma\eta \leq \gamma$, the network has learned to consistently partition the feature space of $\mathcal{X}$. Further, substituting (17) into (11) gives

$$\mathcal{L}_1(\mathcal{N}, \mathcal{X}) \geq \frac{1}{mn} \sum_{i=1}^{n} \left[\frac{1}{(1-\gamma)} \sum_{\forall \mathbf{x} \in \mathcal{C}_i} (\eta(1-\gamma)) + \frac{1}{(\gamma)} \sum_{\forall \mathbf{x} \notin \mathcal{C}_i} (\eta(\gamma)) \right] = \eta. \tag{18}$$

These inferences and Lemma 1 are summarized in Theorem 2 below.

Theorem 2: A conventional FFNN will learn to consistently partition the feature space of a data set $\mathcal{X}$ and the learning process will terminate by satisfying the condition

$$\frac{1}{mn} \sum_{i=1}^{n} \left[\frac{1}{(1-\gamma)} \sum_{\forall \mathbf{x} \in \mathcal{C}_i} (\ \Pi_i \mathcal{N}(\mathbf{x}) - \gamma\ ) + \frac{1}{\gamma} \sum_{\forall \mathbf{x} \notin \mathcal{C}_i} (\ \gamma - \Pi_i \mathcal{N}(\mathbf{x})\ ) \right] \geq \eta > 0, \tag{19}$$

for some $\eta \in [0,1]$ if and only if for $i = 1, 2, \ldots, n$,

$$\Pi_i \mathcal{N}(\mathbf{x}) \begin{cases} > & \gamma + \alpha(1-\gamma) \quad \text{if } \mathbf{x} \in \mathcal{C}_i \\ \leq & \gamma - \alpha\gamma \qquad\quad \text{otherwise,} \end{cases} \tag{20}$$

for some $\alpha \in [0,1]$, with $\alpha \geq \eta$.

Consider that a conventional FFNN is trained to consistently partition the feature space of the data set $\mathcal{X}$ by satisfying the criterion (11). Then, for a given value of η used in (11), the possible values of the response of the ith output unit for both $\mathbf{x} \in \mathcal{C}_i$ and $\mathbf{x} \notin \mathcal{C}_i$ are shown in Figure 1. It is clear from this figure that the greater the value of η, the harder the network is trained and therefore the closer it is to learning the exact classifier function for $\mathcal{X}$. For smaller values of η, the response of the ith unit of the network for $\mathbf{x} \in \mathcal{C}_i$ approaches the response for $\mathbf{x} \notin \mathcal{C}_i$. If $\eta \to 0$, then $\Pi_i \mathcal{N}(\mathbf{x} \in \mathcal{C}_i)$ and $\Pi_i \mathcal{N}(\mathbf{x} \notin \mathcal{C}_i)$ may tend to γ.

3 Properties of Discriminant Functions Learned by FFNN Classifiers

This section studies the characteristics of a conventional FFNN that has learned to consistently partition the feature space of a given data set.

Definition 6: The *decision space γ-partitions* are the functions $\Pi_i \mathcal{N}(\mathbf{x}) = \gamma$, for $i = 1, 2, \ldots, n$ that partition the feature space $\mathcal{X}$ into subsets,

$$\mathcal{D}_i^{(\gamma)} = \{\mathbf{x} \in \mathcal{X} : \Pi_i \mathcal{N}(\mathbf{x}) > \gamma\} \tag{21}$$

and

$$\bar{\mathcal{D}}_i^{(\gamma)} = \{\mathbf{x} \in \mathcal{X} : \Pi_i \mathcal{N}(\mathbf{x}) \leq \gamma\}. \tag{22}$$

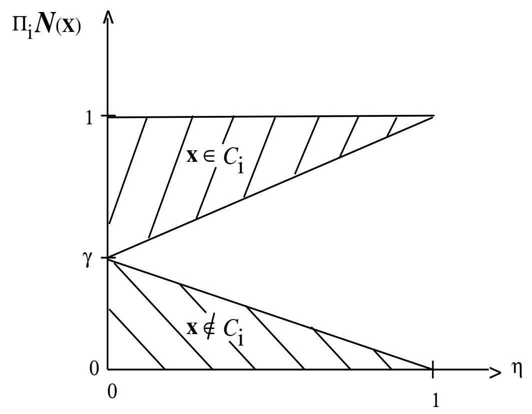

Figure 1: The variation of the range of output values with η.

A conventional FFNN that has been trained to consistently partition the feature space of the data set $\mathcal{X}$ divides the feature space into $\mathcal{D}_i^{(\gamma)}$ and $\bar{\mathcal{D}}_i^{(\gamma)}$ such that $\mathcal{D}_i^{(\gamma)}$ is its best representation of $\mathcal{C}_i \subset \mathcal{X}$, for $i = 1, 2, \ldots, n$. When this network is used as a classifier for the data set $\mathcal{X}$, for every $\mathbf{x}$ falling in the subset $\mathcal{D}_i^{(\gamma)}$, the network has made the decision that $\mathbf{x}$ is in the class $\mathcal{C}_i$.

Definition 7: The *hidden unit λ_j-partition* is the function $g(\mathbf{v}_j^T\mathbf{x} + v_{j0}) = \lambda_j$ that partitions the feature space $\mathcal{X}$ into subsets,

$$\mathcal{G}_j^{(\lambda_j)} = \left\{\mathbf{x} \in \mathcal{X} : g(\mathbf{v}_j^T\mathbf{x} + v_{j0}) > \lambda_j\right\} \tag{23}$$

and

$$\bar{\mathcal{G}}_j^{(\lambda_j)} = \left\{\mathbf{x} \in \mathcal{X} : g(\mathbf{v}_j^T\mathbf{x} + v_{j0}) \leq \lambda_j\right\}. \tag{24}$$

Proposition 5: The hidden unit partitions are linear, that is, the n_i-dimensional surface separating $\mathcal{G}_j^{(\lambda_j)}$ and $\bar{\mathcal{G}}_j^{(\lambda_j)}$ is an hyperplane $\forall \lambda_j$ and for $j = 1, 2, \ldots, n_h$.

Proof: Since g is continuous and monotonically increasing, $g(\mathbf{v}_j^T\mathbf{x} + v_{j0}) = \lambda_j$ implies $\mathbf{v}_j^T\mathbf{x} + v_{j0} = g^{-1}(\lambda_j)$, which is the equation of a n_i-dimensional hyperplane.

Proposition 6: Suppose a conventional FFNN has learned to consistently partition the feature space of the data set $\mathcal{X}$. Then, for $i = 1, 2, \ldots, n$, and every pair of feature vectors $(\mathbf{p}, \mathbf{q})$ such that $\mathbf{p} \in \mathcal{C}_i$ and $\mathbf{q} \notin \mathcal{C}_i$, there exists at least one hidden unit partition between $\mathbf{p}$ and $\mathbf{q}$.

Proof: It needs to be shown that there exists at least one hidden unit λ_r-partition such that $\mathbf{p} \in \mathcal{G}_r^{(\lambda_r)}$ and $\mathbf{q} \in \bar{\mathcal{G}}_r^{(\lambda_r)}$. Suppose for $j = 1, 2, \ldots, n_h$, $\forall \lambda_j$, both $\mathbf{p}$ and $\mathbf{q}$ are either in $\mathcal{G}_j^{(\lambda_j)}$ or $\bar{\mathcal{G}}_j^{(\lambda_j)}$. Then it follows that there exists no λ_j, $j = 1, 2, \ldots, n_h$, such that $g(\mathbf{v}_j^T\mathbf{p} + v_{j0}) > \lambda_j \geq g(\mathbf{v}_j^T\mathbf{q} + v_{j0})$ or $g(\mathbf{v}_j^T\mathbf{p} + v_{j0}) \leq \lambda_j < g(\mathbf{v}_j^T\mathbf{q} + v_{j0})$. This implies that $g(\mathbf{v}_j^T\mathbf{p} + v_{j0}) = g(\mathbf{v}_j^T\mathbf{q} + v_{j0})$, $j = 1, 2, \ldots, n_h$. Multiplying both sides by w_{ij} and summing from $j = 1$ to n_h gives $\sum_{j=1}^{n_h} w_{ij} g(\mathbf{v}_j^T\mathbf{p} + v_{j0}) + w_{i0} = \sum_{j=1}^{n_h} w_{ij} g(\mathbf{v}_j^T\mathbf{q} + v_{j0}) + w_{i0}, \forall i$. This gives $\Pi_i\mathcal{N}(\mathbf{p}) = \Pi_i\mathcal{N}(\mathbf{q}), \forall i$, which implies that there exists no $\gamma \in (0, 1)$ such that $\Pi_i\mathcal{N}(\mathbf{p}) > \gamma \geq \Pi_i\mathcal{N}(\mathbf{q}), \forall i$.

Consider a conventional FFNN that has learned to consistently partition the feature space of the data set $\mathcal{X}$. By Definitions 5 and 6, for every pair of feature vectors $(\mathbf{p}, \mathbf{q})$ such that $\mathbf{p} \in \mathcal{C}_i$ and $\mathbf{q} \notin \mathcal{C}_i$, the network satisfies the condition that $\mathbf{p} \in \mathcal{D}_i^{(\gamma)}$ and $\mathbf{q} \in \bar{\mathcal{D}}_i^{(\gamma)}$. Moreover, according to Proposition 6, the network also satisfies the condition that, for at least one j, $\mathbf{p} \in \mathcal{G}_j^{(\lambda_j)}$ and $\mathbf{q} \in \bar{\mathcal{G}}_j^{(\lambda_j)}$. From Definition 1 and Proposition 5, it may be cursorily concluded that the output space of this conventional FFNN is formed as a linear superposition of the linear partitions of the feature space, with at least one linear partition between each pair of feature vectors belonging to different classes.

Definition 8: Suppose a conventional FFNN has learned to consistently partition the feature space of the data set $\mathcal{X}$. Given a pair of feature vectors $(\mathbf{p}, \mathbf{q})$ such that $\mathbf{p} \in \mathcal{C}_i$ and $\mathbf{q} \notin \mathcal{C}_i$, the *degree of discrimination* δ between $\mathbf{p}$ and $\mathbf{q}$ is defined as

$$\delta = \Pi_i\mathcal{N}(\mathbf{p}) - \Pi_i\mathcal{N}(\mathbf{q}). \tag{25}$$

Clearly, the range of values of δ is the interval $[0, 1]$. If the network has learned the classifier function for $\mathcal{X}$, then $\forall i$ and $\forall (\mathbf{p}, \mathbf{q})$ such that $\mathbf{p} \in \mathcal{C}_i$ and $\mathbf{q} \notin \mathcal{C}_i$, $\delta = 1$.

Proposition 7: The *gradient of the response of the ith unit* of the network, if it exists, can be computed as

$$\nabla_{\mathbf{x}} \Pi_i\mathcal{N}(\mathbf{x}) = \sum_{j=1}^{n_h} w_{ij} \mathbf{v}_j g'(\mathbf{v}_j^T\mathbf{x} + v_{j0}), \tag{26}$$

where g' is the derivative of g.

Proof: Computing the gradient of $\Pi_i\mathcal{N}(\mathbf{x})$ in (1) directly gives the above.

The significance of the gradient of the network response is that its norm given by $\|\nabla_{\mathbf{x}} \Pi_i\mathcal{N}(\mathbf{x})\|$, evaluated at the boundary of a class, gives the rate of change of the response of the ith output unit at the boundary. Therefore, this norm is a measure of the discrimination that the ith output unit of the network has learned to exercise between the feature vectors that belong to the ith class and those that do not. The following theorem clearly shows the manner in which the distance between two feature vectors belonging to two different classes in the given data set influences the gradient of the response at a point "between" the two feature vectors.

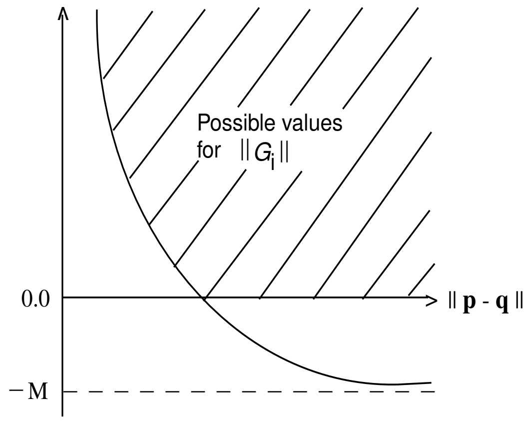

Figure 2: The plot of $\|G_i\|$ as a function of $\|\mathbf{p}-\mathbf{q}\|$.

Lemma 2: Suppose a conventional FFNN has learned to consistently partition the feature space of the data set $\mathcal{X}$. Let $\mathbf{p} \in \mathcal{C}_i$ and $\mathbf{q} \notin \mathcal{C}_i$. Let δ be the amount of discrimination introduced between $\mathbf{p}$ and $\mathbf{q}$ by the ith output unit of the network, and G_i be the average gradient of the response of the ith output unit evaluated at points between the projections of $\mathbf{p}$ and $\mathbf{q}$ along the directions of $\mathbf{v}_j$, $j = 1, 2, \ldots, n_h$. Then the norm of this average gradient of response is given by

$$\|G_i\| \geq c\left(\frac{\delta}{\|\mathbf{p}-\mathbf{q}\|}\right) - M, \tag{27}$$

where c and M are positive constants.

Proof: The proof of Lemma 2 is presented in Appendix D.

Figure 2 plots $\|G_i\|$ as a function of the distance between the vectors $\mathbf{p}$ and $\mathbf{q}$. It is clear from the figure that in order to be able to discriminate between two closely spaced feature vectors belonging to different classes, the gradient of the network response must be correspondingly higher. Suppose a conventional FFNN is trained to consistently partition the feature space of a data set consisting of closely spaced classes. Then, in order to be able to discriminate between feature vectors belonging to different classes but closely spaced, the network response changes rather rapidly from a value greater than γ to a value less than γ, between the feature vectors. The closer the vectors move towards each other, the sharper the transition between the two classes. Two important consequences arise: First, the value of the response at the point of intersection of the response curves for the two classes progressively decreases as the two classes move closer to each other, as shown in Figure 3. Second, in

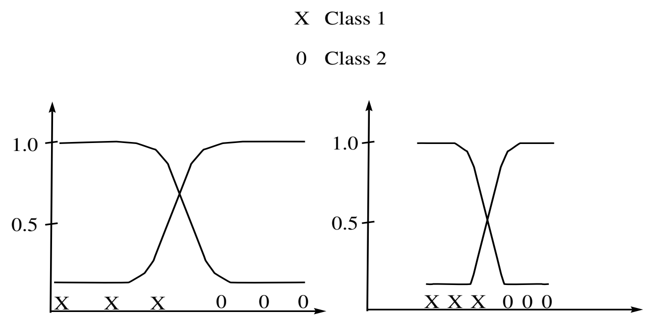

Figure 3: The effect of uncertainty in the feature space on the FFNN response.

the limit as the boundaries of the two classes become contiguous, the network segments the feature space into disjoint regions, very much in conformation with the law of the excluded middle. This is shown in Figure 4.

However, a human interpretation of this data set will be different from the decisions made by the FFNN. Since the two classes are closely spaced, there is maximum uncertainty, ambiguity, and imprecision as to the membership of the feature vectors lying at the boundary between the two classes. The membership of the feature vectors lying near the boundary between the classes should not be very different for these two classes, since they can only be classified as belonging to each class more or less with the same certainty/uncertainty.

This apparent inconsistency between the behavior of the FFNN trained to consistently partition a feature space and a human interpretation of the same feature space arises due to the 'black-box' approach to the supervised training of the FFNN [13]. In an attempt to learn the 'input-output relationship', the FFNN completely ignores all structure and order in the nature and organization of the properties measured and translated into this feature space [13].

In almost all the gradient-descent-based algorithms for learning the internal parameters of the network (synaptic weights), the parameters are updated solely with the aim of training the network to learn the 'input-output relationship', i.e., the function defined on the feature space. Therefore the network fails to learn any inherent structure in the feature space, unless *explicitly trained* to do so [12, 16]. In the following, the norm of the gradient of the network response is related to the hardness of training it has been subjected to.

Suppose a conventional FFNN learns to consistently partition the feature space of the data set $\mathcal{X}$ by satisfying the termination criterion (11) with $\gamma = 1/2$. Let $(\mathbf{p}^{(i)}, \mathbf{q}^{(i)})$ be an ordered pair of feature vectors such that $\mathbf{p}^{(i)} \in \mathcal{C}_i$ and $\mathbf{q}^{(i)} \notin \mathcal{C}_i$. Let $\|\mathbf{p}^{(i)} - \mathbf{q}^{(i)}\|$ be

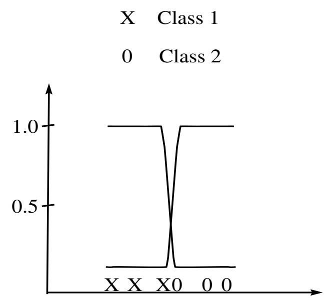

Figure 4: Limiting disjoint partitions of the feature space generated by the FFNN.

their distance of separation. Consider all such ordered pairs consisting of one feature vector lying inside the boundary of the ith class and the other lying outside the boundary. Then, $D_{(i)} = \max_{\mathcal{C}_i} \|\mathbf{p}^{(i)} - \mathbf{q}^{(i)}\|$ is the maximum distance of separation over all such pairs of feature vectors. Further, $D = \max_i D_{(i)}$ is the maximum distance of separation between any two feature vectors lying on either side of a class boundary in the entire data set. The distance D is a measure of the 'closeness' of the classes in the data set, i.e., a measure of the degree of overlapping between feature vectors of different classes in the data set. Let $\|G\|$ be the average of $\|G_i\|$ over all i.

Lemma 3: Suppose a conventional FFNN learns to consistently partition the feature space of the data set $\mathcal{X}$ by satisfying the termination criterion (11) with $\gamma = 1/2$. Then the norm of the average gradient of the network response has the lower bound given by

$$\|G\| \geq c\left(\frac{\eta + (2/n - 1)}{D}\right) - M, \tag{28}$$

where c and M are positive constants.

Proof: The proof of Lemma 3 is presented in Appendix E.

The following inferences can be made from Lemma 3: The number of classes in the data set is at least two, i.e., $n \geq 2$. Consider first the case $n = 2$. Then equation (28) simplifies to give

$$\|G\| \geq c\left(\frac{\eta}{D}\right) - M. \tag{29}$$

Therefore, for a given hardness of training, the closer the two classes are in the given data set, the higher the average slope of the response curves around the class boundaries. Consider a data set consisting of two closely spaced classes. If the network response is desired to fall off at the boundaries gradually, then the network must be trained with a correspondingly low value of η. In the above equation, $\|G\|$ can be made independent of D only by setting $\eta = 0$. In other words, if the data set consists of closely spaced classes and the network is desired to recognize the uncertainty inherent in the feature space, then the network should only be minimally trained or not trained at all ($\eta = 0$)! This shows the basic limitation in the architecture of the conventional FFNNs; *if they are trained with the available sample data, they fail to recognize any uncertainty inherent in the feature space.*

It is often assumed that because the hidden and output units are sigmoidal and can respond continuously over the feature space, the network may respond with valid class membership values that represent a measure of the extent to which a certain feature vector belongs to each class (see, for example, [23]). Clearly, responding with values *between* 0 and 1 can hardly be categorized as 'fuzzy' response when the rate of change of the values is the highest where it should be the lowest.

Consider the general case $n > 2$. Then, as $n \to \infty$, the bound (28) becomes

$$\|G\| \geq c\left(\frac{\eta - 1}{D}\right) - M, \tag{30}$$

which implies that even if the network is hard trained to learn the classifier function (i.e., $\eta = 1$), the slope can still be kept low. However, $\|G\|$ is the norm of the gradient of the response averaged over all the classes and the denominator in (28) is the distance between a feature vector in the ith class and the farthest of the feature vectors not in the ith class. Hence, in the limit $n \to \infty$, the distance between a feature vector in the ith class and the farthest of the feature vectors not in the ith class is also very large. In this situation, (28) implies that the average slope around the boundaries can be very low. A more direct result similar to that in Lemma 2 can be derived as a consequence of Theorem 2.

Theorem 3: Suppose a conventional FFNN learns to consistently partition the feature space of the data set $\mathcal{X}$ by satisfying the termination criterion (11). Let $\mathbf{p} \in \mathcal{C}_i$ and $\mathbf{q} \notin \mathcal{C}_i$. Let G_i be the average of the gradient of the response of the ith output unit evaluated at points between the projections of $\mathbf{p}$ and $\mathbf{q}$ along the directions of $\mathbf{v}_j$, $\forall j$. Then $\|G_i\|$ satisfies the inequality

$$\|G_i\| \geq c\,\frac{\eta}{\|\mathbf{p} - \mathbf{q}\|} - M, \tag{31}$$

where c and M are positive constants.

Proof: Theorem 2 gives for $i = 1, 2, \ldots, n$,

$$\Pi_i \mathcal{N}(\mathbf{q} \notin \mathcal{C}_i) \leq \gamma - \alpha\gamma < \gamma < \gamma + \alpha(1 - \gamma) < \Pi_i \mathcal{N}(\mathbf{p} \in \mathcal{C}_i). \tag{32}$$

This directly implies the inequality,

$$\alpha \leq \Pi_i \mathcal{N}(\mathbf{p}) - \Pi_i \mathcal{N}(\mathbf{q}). \tag{33}$$

The inequality (31) can be obtained by following exactly the same steps as in the proof of Lemma 2 from hereon and noting that, by Theorem 2, $\alpha \geq \eta$.

It is clear from Theorem 3 that if a conventional FFNN is trained at all ($\eta > 0$), then it will create sharp transitions between closely spaced feature vectors belonging to different classes, *even if the hidden and output unit transfer functions are sigmoidal and can respond continuously.* This proves that the ability of the FFNN to estimate class membership profiles from the given sample data is sensitively dependent on the topology of the feature space.

It is also clear from the methodology used to prove Theorem 3 that the lower bound given by the theorem holds for the general case of multilayer perceptrons with any number of hidden layers. Though most of the analysis in this section has been limited to the conventional FFNN (with a single hidden layer), it is clear from the methodology of the proofs and derivations that these results essentially hold for FFNNs with any number of hidden layers.

4 Conclusions

Several attempts have been made to relate the properties of FFNN classifiers to those of statistical classifiers in order to gain insights into the performance of FFNN classifiers [4, 20] For example, it is shown in [20] that training FFNNs to minimize the quadratic error function results in the networks behaving like approximate Bayesian classifiers. However, such an interpretation of the FFNN functionalism relates to the performance of FFNNs *trained in a certain way*, rather than their *inherent ability to perform a certain computational task.* The architecture of the FFNN is not taken into account in this interpretation. When FFNNs are trained to minimize the quadratic error function, the network size (eg., the number of hidden units) governs the type of the minimum achievable. For example, increasing the number of hidden units results in the network learning a higher order polynomial [6]. This directly implies a globally different error surface, and fewer misclassifications.

This paper presented the development of a theoretical study of FFNN classifiers from formal definitions of known and measurable network parameters. The training and learning aspects of the FFNN classifiers were considered first. When a conventional FFNN is trained to function as a classifier, the network learns an approximation to the classifier function (Definition 3). This implies that the network learns to approximate the classifier function with responses ranging between 0 and 1 over the entire feature space. This has been interpreted as the ability of the network to give fuzzy response [23]. The theoretical study presented in this paper proves that the nature of the intermediate response values generated by the FFNN (i.e., response values ranging from 0 to 1) is often reflective of the hardness of training that the FFNN is subjected to. In other words, these intermediate responses could very well be due to the approximation error in the FFNN fit to the classifier function, and may not always correspond to valid membership values (i.e., class membership values consistent with the distribution of data on the feature space).

The influence of the topology of the feature space on the performance of the FFNN classifier was also investigated. Theorem 3 proved the inconsistency of the 'membership values' generated by FFNNs by showing that they create sharp transitions between closely

spaced feature vectors belonging to overlapping classes. In summary, FFNNs trained in the conventional manner to function as classifiers for overlapping classes of data may not be good indicators of the uncertainty inherent in the data.

Finally the theoretical study presented in this paper indicates that the decision space is formed as a linear superposition of the hidden unit partitions. Also, the hidden unit partitions are linear because the hidden unit transfer functions are sigmoidal. Therefore, if the hidden unit transfer functions are modified to allow brief discontinuities, then the feature space partitions generated by the network will transition in a graded manner, indicative of the uncertainty in the feature space. One possible modification in the network architecture that might bring about the desired changes in the network response led to the development of inherently fuzzy feedforward neural networks, known as quantum neural networks (QNNs) [11, 18, 19]. In addition to their function approximation capabilities, QNNs have been shown to be capable of identifying and quantifying the uncertainty inherent in the training data. More specifically, QNNs can identify overlapping between classes due to their capacity of approximating any arbitrary membership profile up to any degree of accuracy. QNNs have been used by several research teams around the world in a number of practical applications, which include the recognition of handwritten digits [25], the identification and removal of bird-contaminated data from the recordings of a 1290-MHz wind profiler [9, 14], and the detection of epileptic seizure segments from neonatal EEG [10].

Appendix A

Proof of Proposition 3

Proof for Only If: Since $\mathcal{N}$ has learned to consistently partition the feature space of $\mathcal{X}$, $\Pi_i \mathcal{N}(\mathbf{x}) \in [0,1]$, $\forall i$ and $\mathbf{x}$. Thus, $\Pi_i \mathcal{N}(\mathbf{x}) - \gamma \leq 1-\gamma$, $\forall i$ and $\mathbf{x}$, and $\gamma - \Pi_i \mathcal{N}(\mathbf{x}) \leq \gamma$, $\forall i$ and $\mathbf{x}$. Substituting these bounds into the expression for $\mathcal{L}_1(\mathcal{N}, \mathcal{X})$ gives

$$\mathcal{L}_1(\mathcal{N}, \mathcal{X}) \leq \frac{1}{mn} \sum_{i=1}^{n} \left[\frac{1}{(1-\gamma)} \sum_{\forall \mathbf{x} \in \mathcal{C}_i} (1-\gamma) + \frac{1}{\gamma} \sum_{\forall \mathbf{x} \notin \mathcal{C}_i} (\gamma) \right] = 1. \tag{A1}$$

Combining (11) and (A1) gives

$$\mathcal{L}_1(\mathcal{N}, \mathcal{X}) = 1. \tag{A2}$$

Combining (A2) with the definition of $\mathcal{L}_1(\mathcal{N}, \mathcal{X})$ in (11) gives

$$\sum_{i=1}^{n} \left[\frac{1}{(1-\gamma)} \sum_{\forall \mathbf{x} \in \mathcal{C}_i} (\ 1 - \Pi_i \mathcal{N}(\mathbf{x})\ ) + \frac{1}{\gamma} \sum_{\forall \mathbf{x} \notin \mathcal{C}_i} (\ \Pi_i \mathcal{N}(\mathbf{x})\ ) \right] = 0. \tag{A3}$$

But, as noted above, $\Pi_i \mathcal{N}(\mathbf{x}) \in [0,1]$, $\forall i$ and $\mathbf{x}$. Thus, $1 - \Pi_i \mathcal{N}(\mathbf{x}) \geq 0$, and $\Pi_i \mathcal{N}(\mathbf{x}) \geq 0$. Hence equation (A3) will be satisfied iff for $i = 1, 2, \ldots, n$,

$$\Pi_i \mathcal{N}(\mathbf{x}) = \begin{cases} 1 \text{ if } \mathbf{x} \in \mathcal{C}_i \\ 0 \text{ otherwise.} \end{cases} \tag{A4}$$

Proof for If: The expression for $\mathcal{L}_1 = \mathcal{L}_1(\mathcal{N}, \mathcal{X})$ can be rewritten as

$$\begin{aligned}\mathcal{L}_1 &= \frac{1}{mn}\sum_{i=1}^{n}\left[\frac{1}{(1-\gamma)}\sum_{\forall \mathbf{x}\in\mathcal{C}_i}(\ \Pi_i F(\mathbf{x}) - \gamma\ ) + \frac{1}{\gamma}\sum_{\forall \mathbf{x}\notin\mathcal{C}_i}(\ \gamma - \Pi_i F(\mathbf{x})\ )\right] \\ &-\frac{1}{mn}\sum_{i=1}^{n}\left[\frac{1}{(1-\gamma)}\sum_{\forall \mathbf{x}\in\mathcal{C}_i}(\ \Pi_i F(\mathbf{x}) - \Pi_i\mathcal{N}(\mathbf{x})\ ) + \frac{1}{\gamma}\sum_{\forall \mathbf{x}\notin\mathcal{C}_i}(\ \Pi_i\mathcal{N}(\mathbf{x}) - \Pi_i F(\mathbf{x})\ )\right]\end{aligned} \quad (A5)$$

Since $\mathcal{N}$ has learned F, given an $\varepsilon > 0$, there exists $\mathbf{V}, g, \mathbf{W}$ such that

$$-(1-\gamma)\varepsilon \ < \Pi_i F(\mathbf{x}) - \Pi_i\mathcal{N}(\mathbf{V}, g, \mathbf{W};\, \mathbf{x}) < \ (1-\gamma)\varepsilon \quad (A6)$$

and

$$-\gamma\varepsilon \ < \Pi_i F(\mathbf{x}) - \Pi_i\mathcal{N}(\mathbf{V}, g, \mathbf{W};\, \mathbf{x}) < \ \gamma\varepsilon. \quad (A7)$$

Substituting these conditions into the expansion for $\mathcal{L}_1(\mathcal{N}, \mathcal{X})$, the following inequality is obtained:

$$\begin{aligned}\mathcal{L}_1(\mathcal{N}, \mathcal{X}) &> \frac{1}{mn}\sum_{i=1}^{n}\left[\frac{1}{(1-\gamma)}\sum_{\forall \mathbf{x}\in\mathcal{C}_i}(1-\gamma) + \frac{1}{\gamma}\sum_{\forall \mathbf{x}\notin\mathcal{C}_i}\gamma\right] \\ &\quad -\frac{1}{mn}\sum_{i=1}^{n}\left[\frac{1}{(1-\gamma)}\sum_{\forall \mathbf{x}\in\mathcal{C}_i}\varepsilon(1-\gamma) + \frac{1}{\gamma}\sum_{\forall \mathbf{x}\notin\mathcal{C}_i}\varepsilon\gamma\right] \\ &> 1-\varepsilon.\end{aligned} \quad (A8)$$

Since the above inequality is satisfied for every $\varepsilon > 0$, it directly follows that

$$\mathcal{L}_1(\mathcal{N}, \mathcal{X}) \geq 1. \quad (A9)$$

This completes the proof of Proposition 3.

Appendix B

Proof of Proposition 4

Proof for Only If: Since $\mathcal{N}$ has learned to consistently partition the feature space of $\mathcal{X}$, $\Pi_i\mathcal{N}(\mathbf{x}) \in [0, 1]$, $\forall i$. This implies the condition that $\Pi_i\mathcal{N}(\mathbf{x}) - \Pi_j\mathcal{N}(\mathbf{x}) \leq 1$, $\forall i, j$. Substituting the above condition into the expression for $\mathcal{L}_2(\mathcal{N}, \mathcal{X})$ gives

$$\frac{1}{m(n-1)}\sum_{i=1}^{n}\sum_{\forall \mathbf{x}\in\mathcal{C}_i}\sum_{\forall j\neq i}(\ \Pi_i\mathcal{N}(\mathbf{x}) - \Pi_j\mathcal{N}(\mathbf{x})\ ) \leq \frac{1}{m(n-1)}\sum_{i=1}^{n}\sum_{\forall \mathbf{x}\in\mathcal{C}_i}\sum_{\forall j\neq i} 1 = 1. \quad (B1)$$

Comparing (12) with (B1) gives

$$\frac{1}{m(n-1)}\sum_{i=1}^{n}\sum_{\forall \mathbf{x}\in\mathcal{C}_i}\sum_{\forall j\neq i}(\ \Pi_i\mathcal{N}(\mathbf{x}) - \Pi_j\mathcal{N}(\mathbf{x})\ ) = 1. \quad (B2)$$

It can be verified that (B2) gives

$$\sum_{i=1}^{n}\sum_{\forall \mathbf{x}\in\mathcal{C}_i}\sum_{\forall j\neq i}\{1-(\ \Pi_i\mathcal{N}(\mathbf{x})-\Pi_j\mathcal{N}(\mathbf{x})\ )\}=0. \tag{B3}$$

But, as noted above, $\Pi_i\mathcal{N}(\mathbf{x})-\Pi_j\mathcal{N}(\mathbf{x})\leq 1,\ \ \forall i,j$, which implies

$$1-(\Pi_i\mathcal{N}(\mathbf{x})-\Pi_j\mathcal{N}(\mathbf{x}))\geq 0\ \ \forall i,j. \tag{B4}$$

Therefore, equation (B2) will be satisfied iff for each i and $\forall j\neq i$, $\Pi_i\mathcal{N}(\mathbf{x})-\Pi_j\mathcal{N}(\mathbf{x})=1$, whenever $\mathbf{x}\in\mathcal{C}_i$. This implies

$$\Pi_i\mathcal{N}(\mathbf{x})=\begin{cases}1 \text{ if } \mathbf{x}\in\mathcal{C}_i\\ 0 \text{ otherwise.}\end{cases} \tag{B5}$$

Proof for If: The expression for $\mathcal{L}_2(\mathcal{N},\mathcal{X})$ may be rewritten as

$$\begin{aligned}\mathcal{L}_2(\mathcal{N},\mathcal{X}) \;=\;& \frac{1}{m(n-1)}\sum_{i=1}^{n}\sum_{\forall \mathbf{x}\in\mathcal{C}_i}\sum_{\forall j\neq i}(\ \Pi_iF(\mathbf{x})-\Pi_j\mathcal{N}(\mathbf{x})\ )\\ &-\frac{1}{m(n-1)}\sum_{i=1}^{n}\sum_{\forall \mathbf{x}\in\mathcal{C}_i}\sum_{\forall j\neq i}(\ \Pi_iF(\mathbf{x})-\Pi_i\mathcal{N}(\mathbf{x})\ ).\end{aligned} \tag{B6}$$

Since $\mathcal{N}$ has learned F, given an $\varepsilon>0$, there exists $\mathbf{V},g,\mathbf{W}$ such that

$$1-\frac{\varepsilon}{2}<\frac{1}{m(n-1)}\sum_{i=1}^{n}\sum_{\forall \mathbf{x}\in\mathcal{C}_i}\sum_{\forall j\neq i}(\ \Pi_iF(\mathbf{x})-\Pi_j\mathcal{N}(\mathbf{x})\ )<1+\frac{\varepsilon}{2} \tag{B7}$$

and

$$-\frac{\varepsilon}{2}<\frac{1}{m(n-1)}\sum_{i=1}^{n}\sum_{\forall \mathbf{x}\in\mathcal{C}_i}\sum_{\forall j\neq i}(\ \Pi_iF(\mathbf{x})-\Pi_i\mathcal{N}(\mathbf{x})\ )<\frac{\varepsilon}{2}. \tag{B8}$$

Substituting the above conditions into the expansion for $\mathcal{L}_2(\mathcal{N},\mathcal{X})$ gives

$$\mathcal{L}_2(\mathcal{N},\mathcal{X})>1-\varepsilon. \tag{B9}$$

Since the above inequality is true for every $\varepsilon>0$, it directly follows that

$$\mathcal{L}_2(\mathcal{N},\mathcal{X})\geq 1. \tag{B10}$$

Appendix C

Proof of Lemma 1

For $i=1,2,\ldots,n$, if $\mathbf{x}\in\mathcal{C}_i$, then $\Pi_i\mathcal{N}(\mathbf{x})>\gamma$, or, $\Pi_i\mathcal{N}(\mathbf{x})-\gamma>0$. Set

$$\beta_1=\min_i\min_{\forall \mathbf{x}\in\mathcal{C}_i}\left\{\frac{\Pi_i\mathcal{N}(\mathbf{x})-\gamma}{2}\right\}. \tag{C1}$$

For $i = 1, 2, \ldots, n$, if $\mathbf{x} \notin \mathcal{C}_i$, then $\Pi_i \mathcal{N}(\mathbf{x}) \leq \gamma$, or, $\gamma - \Pi_i \mathcal{N}(\mathbf{x}) \geq 0$. Set

$$\beta_2 = \min_i \min_{\forall \mathbf{x} \in \mathcal{C}_i} \left\{ \frac{\gamma - \Pi_i \mathcal{N}(\mathbf{x})}{2} \right\}. \tag{C2}$$

Now set $\alpha = \min\{\beta_1, \beta_2\}$. Clearly, $\beta_1 > 0$ and $\beta_2 \geq 0$. Furthermore, it is evident that

$$1 > \frac{1-\gamma}{2} > \left\{ \frac{\Pi_i \mathcal{N}(\mathbf{x}) - \gamma}{2} \right\} \tag{C3}$$

and

$$1 > \frac{\gamma}{2} > \left\{ \frac{\gamma - \Pi_i \mathcal{N}(\mathbf{x})}{2} \right\}. \tag{C4}$$

Therefore, $1 \geq \alpha \geq 0$. Since $\beta_1 \geq \min\{\beta_1, \beta_2\} = \alpha$, for $i = 1, 2, \ldots, n$, if $\mathbf{x} \in \mathcal{C}_i$, then it follows that $\Pi_i \mathcal{N}(\mathbf{x}) - \gamma > \beta_1 \geq \alpha$, which implies $\Pi_i \mathcal{N}(\mathbf{x}) > \gamma + \alpha \geq \gamma + \alpha(1-\gamma)$. Similarly, since $\beta_2 \geq \min\{\beta_1, \beta_2\} = \alpha$, for $i = 1, 2, \ldots, n$, if $\mathbf{x} \notin \mathcal{C}_i$, then it follows that $\gamma - \Pi_i \mathcal{N}(\mathbf{x}) \geq \beta_2 \geq \alpha \geq \alpha\gamma$, which implies $\Pi_i \mathcal{N}(\mathbf{x}) \leq \gamma - \alpha\gamma$. This completes the proof for the first part of the Lemma.

If for $i = 1, 2, \ldots, n$,

$$\Pi_i \mathcal{N}(\mathbf{x}) \begin{cases} > & \gamma + \alpha(1-\gamma) \quad \text{if } \mathbf{x} \in \mathcal{C}_i \\ \leq & \gamma - \alpha\gamma \quad \text{otherwise,} \end{cases} \tag{C5}$$

then it follows that

$$\mathcal{L}_2(\mathcal{N}, \mathcal{X}) \geq \frac{1}{mn} \sum_{i=1}^{n} \left[\frac{1}{(1-\gamma)} \sum_{\forall \mathbf{x} \in \mathcal{C}_i} \alpha(1-\gamma) + \frac{1}{\gamma} \sum_{\forall \mathbf{x} \notin \mathcal{C}_i} \alpha\gamma \right] = \alpha. \tag{C6}$$

This completes the proof of Lemma 1.

Appendix D

Proof of Lemma 2

Definition 9 gives

$$\begin{aligned} \delta &= \Pi_i \mathcal{N}(\mathbf{p}) - \Pi_i \mathcal{N}(\mathbf{q}) \\ &= \sum_{j=1}^{n_h} w_{ij} \left(g(\mathbf{v}_j^T \mathbf{p} + v_{j0}) - g(\mathbf{v}_j^T \mathbf{q} + v_{j0}) \right) \\ &= \sum_{j=1}^{n_h} w_{ij} \left(\frac{g(\mathbf{v}_j^T \mathbf{p} + v_{j0}) - g(\mathbf{v}_j^T \mathbf{q} + v_{j0})}{\mathbf{v}_j^T (\mathbf{p} - \mathbf{q})} \right) \mathbf{v}_j^T (\mathbf{p} - \mathbf{q}). \end{aligned} \tag{D1}$$

Applying the mean value theorem [21] to the right side of (D1) gives

$$\delta = \sum_{j=1}^{n_h} w_{ij} g'(\mathbf{v}_j^T \mathbf{z}_j + v_{j0}) \mathbf{v}_j^T (\mathbf{p} - \mathbf{q}) = (\mathbf{p} - \mathbf{q})^T \sum_{j=1}^{n_h} w_{ij} \mathbf{v}_j g'(\mathbf{v}_j^T \mathbf{z}_j + v_{j0}), \tag{D2}$$

where $\mathbf{z}_k$, $k = 1, 2, \ldots, n_h$, are vectors such that $\mathbf{v}_j^T\mathbf{q} \le \mathbf{v}_j^T\mathbf{z}_j \le \mathbf{v}_j^T\mathbf{p}$. It is assumed without loss of generality that $\mathbf{v}_j^T\mathbf{q} \le \mathbf{v}_j^T\mathbf{p}$. Let the sum $\sum_{j=1}^{n_h} w_{ij}\mathbf{v}_j g'(\mathbf{v}_j^T\mathbf{z}_j + v_{j0})$ be denoted by $\mathbf{s}$. Then, (D2) gives

$$\delta \le \|\mathbf{p} - \mathbf{q}\|\|\mathbf{s}\|, \tag{D3}$$

by the Cauchy inequality [21]. Let $\nabla_{\mathbf{x}}\Pi_i\mathcal{N}(\mathbf{x})|_{\mathbf{x}=\mathbf{z}_k}$ be the gradient of the response of the ith output unit evaluated at $\mathbf{z}_k$. Computing the average of this gradient over all $\mathbf{z}_k$, $k = 1, 2, \ldots, n_h$, gives

$$\begin{aligned} G_i &= \frac{1}{n_h}\sum_{k=1}^{n_h} \nabla_{\mathbf{x}}\Pi_i\mathcal{N}(\mathbf{x})|_{\mathbf{x}=\mathbf{z}_k} \\ &= \frac{1}{n_h}\sum_{j=1}^{n_h} w_{ij}\mathbf{v}_j g'(\mathbf{v}_j^T\mathbf{z}_j + v_{j0}) + \frac{1}{n_h}\sum_{j=1}^{n_h}\sum_{k\ne j} w_{ij}\mathbf{v}_j g'(\mathbf{v}_j^T\mathbf{z}_k + v_{j0}) \\ &= \frac{1}{n_h}(\mathbf{s} + \mathbf{b}), \end{aligned} \tag{D4}$$

where $\mathbf{b} = \sum_{j=1}^{n_h}\sum_{k\ne j} w_{ij}\mathbf{v}_j g'(\mathbf{v}_j^T\mathbf{z}_k + v_{j0})$. The above equation gives $\mathbf{s} = n_h G_i - \mathbf{b}$, which implies

$$\|\mathbf{s}\| \le n_h\|G_i\| + \|\mathbf{b}\|. \tag{D5}$$

Moreover, since g is sigmoidal, g' is bounded and since the weights are also bounded, $\|\mathbf{b}\|$ is also bounded, i.e.,

$$\|\mathbf{b}\| \le \sum_{j=1}^{n_h}\sum_{k\ne j} |w_{ij}|\ \|\mathbf{v}_j\|\ |g'(\mathbf{v}_j^T\mathbf{z}_k + v_{j0})| \le M', \tag{D6}$$

where M' is some positive constant. Combining (D6), (D5) and (D3) gives (27) with $c = 1/n_h$ and $M = M'/n_h$.

Appendix E

Proof of Lemma 3

Substituting $\gamma = 1/2$ in (11) gives

$$\begin{aligned} \eta &\le \frac{2}{mn}\sum_{i=1}^{n}\left[\sum_{\mathbf{x}\in\mathcal{C}_i}\left(\Pi_i\mathcal{N}(\mathbf{x}) - \frac{1}{2}\right) + \sum_{\mathbf{x}\notin\mathcal{C}_i}\left(\frac{1}{2} - \Pi_i\mathcal{N}(\mathbf{x})\right)\right] \\ &= \frac{2}{mn}\sum_{i=1}^{n}\left[\sum_{\mathbf{x}\in\mathcal{C}_i}\Pi_i\mathcal{N}(\mathbf{x}) - \sum_{\mathbf{x}\in\mathcal{C}_i}\frac{1}{2} - \sum_{\mathbf{x}\notin\mathcal{C}_i}\Pi_i\mathcal{N}(\mathbf{x}) + \sum_{\mathbf{x}\notin\mathcal{C}_i}\frac{1}{2}\right] \\ &= \frac{2}{mn}\sum_{i=1}^{n}\left[\sum_{\mathbf{x}\in\mathcal{C}_i}\Pi_i\mathcal{N}(\mathbf{x}) - \sum_{\mathbf{x}\notin\mathcal{C}_i}\Pi_i\mathcal{N}(\mathbf{x}) - \frac{1}{2}\mathrm{Card}_i + \frac{1}{2}\mathrm{Card}'_i\right]. \end{aligned} \tag{E1}$$

Assume without loss of generality that, for the two-class problem, $\mathrm{Card}'_i \ge \mathrm{Card}_i$ and for the multi-class problem, for each i, $\mathrm{Card}'_i \ge \mathrm{Card}_i$. Since $\Pi_i\mathcal{N}(\mathbf{x})$ is nonnegative, leaving

out a few terms in $\sum_{\mathbf{x}\notin\mathcal{C}_i} \Pi_i\mathcal{N}(\mathbf{x})$ preserves the inequality. Hence equation (E1) gives

$$\begin{aligned}\eta \;\leq\; & \frac{2}{mn}\sum_{i=1}^{n}\left[\sum_{(\mathbf{p}^{(i)},\mathbf{q}^{(i)})}\left\{\Pi_i\mathcal{N}(\mathbf{p}^{(i)}) - \Pi_i\mathcal{N}(\mathbf{q}^{(i)})\right\}\right] \\ & +\frac{2}{mn}\left[-\sum_{i=1}^{n}\mathrm{Card}_i + \frac{1}{2}\sum_{i=1}^{n}(\mathrm{Card}_i + \mathrm{Card}'_i)\right]. \end{aligned} \tag{E2}$$

Since $\sum_{i=1}^{n}\mathrm{Card}_i \geq m$ due to overlapping, the above equation simplifies to

$$\eta + \left(\frac{2}{n} - 1\right) \leq \frac{2}{mn}\sum_{i=1}^{n}\left[\sum_{(\mathbf{p}^{(i)},\mathbf{q}^{(i)})}\left\{\Pi_i\mathcal{N}(\mathbf{p}^{(i)}) - \Pi_i\mathcal{N}(\mathbf{q}^{(i)})\right\}\right]. \tag{E3}$$

As in the proof of Lemma 2, $\left\{\Pi_i\mathcal{N}(\mathbf{p}^{(i)}) - \Pi_i\mathcal{N}(\mathbf{q}^{(i)})\right\}$ can be rewritten in terms of $\|G_i\|$ as

$$\begin{aligned}\eta + \left(\frac{2}{n} - 1\right) \;&\leq\; \frac{2}{mn}\sum_{i=1}^{n}\left[\sum_{(\mathbf{p}^{(i)},\mathbf{q}^{(i)})}(\|G_i\| - M')\,\|\mathbf{p}-\mathbf{q}\|\right] \\ &\leq\; \frac{2}{mn}\sum_{i=1}^{n}\left[\sum_{(\mathbf{p}^{(i)},\mathbf{q}^{(i)})}(\|G_i\| - M')\right]D, \end{aligned} \tag{E4}$$

where M' is a positive constant. The lower bound (28) can easily be obtained from the above inequality.

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Exponential operators for solving evolution problems with degeneration

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Abstract

We introduce suitable exponential operators and use the multi-dimensional polynomials considered by Hermite, and subsequently studied by P. Appell and J. Kampé de Fériet, H.W. Gould – A.T. Hopper, G. Dattoli et al. in order to obtain explicit solutions of classical and generalized PDE problems in the half-plane $y > 0$, connecting systems with equations, and including some problems with degeneration on the x-axis.

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1 Introduction

In preceding articles [6], [15], [16] the multi-dimensional polynomials considered by Hermite, and subsequently studied by P. Appell and J. Kampé de Fériet [1], H.W. Gould and A.T. Hopper [12], [18] G. Dattoli et al. [7], was stressed in order to obtain explicit solutions of all the classical (parabolic, hyperbolic and elliptic) BVP in the half-plane and some of their generalizations.

For shortness, in the following, we will use the abbreviation H-KdF for denoting the Hermite-Kampé de Fériet polynomials.

According to our results, the two-dimensional H-KdF polynomials appear as the natural tools for the solution of all these problems.

In the present article, we apply the same technique for finding solutions of classical and generalized PDE problems in the half-plane $y > 0$, connecting systems with equations, and including some problems with degeneration on the x-axis.

In a forthcoming article, we will consider similar problems for pseudo-classical systems, by using pseudo-hyperbolic or pseudo-circular functions of the derivative operator (see [5], [6], [16]).

2 Hermite-Kampé de Fériet polynomials

2.1 Definitions

We recall the definitions of the H-KdF polynomials, starting from the two-dimensional case.

Put $D := \frac{d}{dx}$, and consider the shift operator

$$e^{yD} f(x) = f(x+y) = \sum_{n=0}^{\infty} \frac{y^n}{n!}\ f^{(n)}(x), \tag{2.1}$$

(see e.g. [19], p. 171), the second equation being meaningful for analytic functions. Note that,

- if $f(x) = x^m$, then $e^{yD} x^m = (x+y)^m$;
- if $f(x) = \sum_{m=0}^{\infty} a_m x^m$, then $e^{yD} f(x) = \sum_{m=0}^{\infty} a_m (x+y)^m$.

Definition 2.1 *The Hermite polynomials in two variables* $H_m^{(1)}(x,y)$ *are then defined by*

$$H_m^{(1)}(x,y) := (x+y)^m. \tag{2.2}$$

Consequently,

- if $f(x) = \sum_{m=0}^{\infty} a_m x^m$, then

$$e^{yD} f(x) = \sum_{m=0}^{\infty} a_m H_m^{(1)}(x,y). \tag{2.3}$$

Consider now the exponential containing the second derivative, defined for an analytic function f, as follows:

$$e^{yD^2} f(x) = \sum_{n=0}^{\infty} \frac{y^n}{n!}\ f^{(2n)}(x). \tag{2.4}$$

Note that

- if $f(x) = x^m$, then for $n = 0, 1, \ldots, \left[\frac{m}{2}\right]$ we can write:
  $D^{2n} x^m = m(m-1)\cdots(m-2n+1)x^{m-2n} = \frac{m!}{(m-2n)!}\ x^{m-2n}$ and therefore

$$e^{yD^2} x^m = \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{y^n}{n!} \frac{m!}{(m-2n)!}\ x^{m-2n} \tag{2.5}$$

- if $f(x) = \sum_{m=0}^{\infty} a_m x^m$, then $e^{yD^2} f(x) = \sum_{m=0}^{\infty} a_m H_m^{(2)}(x, y)$.

Definition 2.2 *The H-KdF polynomials in two variables* $H_m^{(2)}(x, y)$ *are then defined by*

$$H_m^{(2)}(x, y) := m! \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{y^n x^{m-2n}}{n!(m-2n)!} \tag{2.6}$$

Considering, in general, the exponential raised to the j-th derivative we have:

$$e^{yD^j} f(x) = \sum_{n=0}^{\infty} \frac{y^n}{n!}\ f^{(jn)}(x) \tag{2.7}$$

and therefore:

- if $f(x) = x^m$, then for $n = 0, 1, \ldots, \left[\frac{m}{j}\right]$ it follows:
  $D^{jn} x^m = m(m-1)\cdots(m-jn+1)x^{m-jn} = \frac{m!}{(m-jn)!}\ x^{m-jn}$ so that

$$e^{yD^j} x^m = \sum_{n=0}^{\left[\frac{m}{j}\right]} \frac{y^n}{n!} \frac{m!}{(m-jn)!}\ x^{m-jn} \tag{2.8}$$

- if $f(x) = \sum_{m=0}^{\infty} a_m x^m$, then $e^{yD^j} f(x) = \sum_{m=0}^{\infty} a_m H_m^{(j)}(x, y)$,

Definition 2.3 *The H-KdF polynomials in two variables* $H_m^{(j)}(x, y)$ *are then defined by*

$$H_m^{(j)}(x, y) := m! \sum_{n=0}^{\left[\frac{m}{j}\right]} \frac{y^n x^{m-jn}}{n!(m-jn)!} \tag{2.9}$$

2.2 Properties

In a number of articles by G. Dattoli et al., (see e.g. [7], [9], [8]), by using the so called *monomiality principle*, the following properties for the two-variable H-KdF polynomials $H_m^{(j)}(x,y), \quad j \geq 2$ have been recovered (the case when $j = 1$ reducing to results about simple powers).

- Operational definition

$$H_m^{(j)}(x,y) = e^{y \frac{\partial^j}{\partial x^j}} x^m = \left(x + jy \frac{\partial^{j-1}}{\partial x^{j-1}} \right)^m (1). \tag{2.10}$$

- Generating function

$$\sum_{n=0}^{\infty} H_n^{(j)}(x,y) \frac{t^n}{n!} = e^{xt+yt^j}. \tag{2.11}$$

In the case when $j = 2$, (see [20]), the H-KdF polynomials $H_m^{(2)}(x,y)$ admit the following

- Integral representation

$$H_m^{(2)}(x,y) = \frac{1}{2\sqrt{\pi y}} \int_{-\infty}^{+\infty} \xi^m e^{-\frac{(x-\xi)^2}{4y}} d\xi, \tag{2.12}$$

which is a particular case of the so called Gauss-Weierstrass (or Poisson) transform.

For the case when $j > 2$ see [13], [14].

3 Definitions of hyperbolic and circular functions of the derivative operator

The two-variable H-KdF polynomials allow us to define, in a constructive way, the hyperbolic and circular functions of the derivative operator. As a matter of fact, for any analytic function q we put, by definition:

$$\cosh(y^\ell D^j)q(x) := \sum_{n=0}^{\infty} \frac{y^{2\ell n} D^{2jn}}{(2n)!} q(x) = \sum_{n=0}^{\infty} \frac{y^{2\ell n}}{(2n)!} q^{(2jn)}(x), \tag{3.1}$$

provided that the last series is convergent in a non-trivial set.

Then, if $q(x) = x^m$, and $n = 0, 1, ..., \left[\frac{m}{2j}\right]$, since: $D^{2jn} x^m = \frac{m!}{(m-2jn)!} x^{m-2jn}$, we can write:

$$\cosh(y^\ell D^j) x^m = m! \sum_{n=0}^{\left[\frac{m}{2j}\right]} \frac{y^{2\ell n} x^{m-2jn}}{(2n)!(m-2jn)!} =: \mathcal{K}_m^{(j)}(x, y^\ell) \tag{3.2}$$

and, in general, if $q(x) = \sum_{m=0}^{\infty} a_m x^m$:

$$\cosh(y^\ell D^j)q(x) = \sum_{m=0}^{\infty} a_m \mathcal{K}_m^{(j)}(x, y^\ell), \tag{3.3}$$

where

$$\mathcal{K}_m^{(j)}(x, y^\ell) = \mathcal{E}_{y^\ell}\left(H_m^{(j)}(x, y^\ell)\right) \tag{3.4}$$

and $\mathcal{E}_{y^\ell}(\cdot)$ denotes the *even part*, with respect to the y^ℓ variable, of the considered H-KdF polynomial.

Proceeding in an analogous way, we define

$$\sinh(y^\ell D^j)q(x) := \sum_{n=0}^{\infty} \frac{y^{\ell(2n+1)} D^{j(2n+1)}}{(2n+1)!} q(x) = \sum_{n=0}^{\infty} \frac{y^{\ell(2n+1)}}{(2n+1)!} q^{j(2n+1)}(x), \tag{3.5}$$

assuming the convergence of the last series in a non-trivial set.

Then, if $q(x) = x^m$, and $n = 0, 1, ..., \left[\frac{m-j}{2j}\right]$, since: $D^{j(2n+1)} x^m = \frac{m!}{(m-2jn-j)!} x^{m-2jn-j}$, we can write:

$$\sinh(y^\ell D^j)x^m = m! \sum_{n=0}^{\left[\frac{m-j}{2j}\right]} \frac{y^{\ell(2n+1)} x^{m-2jn-j}}{(2n+1)!(m-2jn-j)!} =: \mathcal{S}_m^{(j)}(x, y^\ell) \tag{3.6}$$

and, in general, if $q(x) = \sum_{m=0}^{\infty} a_m x^m$:

$$\sinh(y^\ell D^j)q(x) = \sum_{m=0}^{\infty} a_m \mathcal{S}_m^{(j)}(x, y^\ell) \tag{3.7}$$

where

$$\mathcal{S}_m^{(j)}(x, y^\ell) = \mathcal{O}_{y^\ell}\left(H_m^{(j)}(x, y^\ell)\right) \tag{3.8}$$

and $\mathcal{O}_{y^\ell}(\cdot)$ denotes the *odd part*, with respect to y^ℓ, of the considered H-KdF polynomial.

For circular functions the above formulas become:

$$\cos(y^\ell D^j)q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n\ell} D^{2jn}}{(2n)!} q(x) = \sum_{n=0}^{\infty} \frac{(iy^\ell)^{2n}}{(2n)!} q^{(2jn)}(x), \tag{3.9}$$

$$\cos(y^\ell D^j)q(x) = \sum_{n=0}^{\infty} a_m \mathcal{K}_m^{(j)}(x, iy^\ell) = \cosh(iy^\ell D^j)q(x), \tag{3.10}$$

$$\sin(y^\ell D^j)q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{\ell(2n+1)} D^{j(2n+1)}}{(2n+1)!} q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{\ell(2n+1)}}{(2n+1)!} q^{j(2n+1)}(x), \tag{3.11}$$

$$\sin(y^\ell D^j)q(x) = \frac{1}{i}\sum_{m=0}^{\infty} a_m \mathcal{S}_m^{(j)}(x, iy^\ell) = \frac{1}{i}\sinh(iy^\ell D^j)q(x). \tag{3.12}$$

The properties of the hyperbolic and circular functions of the derivative operator, and the relevant generalization to the pseudo-hyperbolic and pseudo-circular functions, was considered in a separate article [5].

We recall here only the definition of the pseudo-hyperbolic and pseudo-circular functions with respect to the cyclic group of order r:

$$f_h(x) := \Pi_{[h,r]}e^x, \qquad g_h(x) := \sigma_0^{-h} f_h(\sigma_0 x), \qquad (h = 0, 1, \ldots, r-1), \tag{3.13}$$

where

$$f_h(z) := \Pi_{[h,r]}e^z := \sum_{n=0}^{\infty} \frac{z^{rn+h}}{(rn+h)!}, \tag{3.14}$$

and σ_0 denotes any complex r-th root of the number -1.

Remark 3.1 If $j = 1$, then by Taylor's theorem we find:

$$\cosh(y^\ell D)q(x) = \frac{1}{2}\left[q(x+y^\ell) + q(x-y^\ell)\right]$$

$$\sinh(y^\ell D)q(x) = \frac{1}{2}\left[q(x+y^\ell) - q(x-y^\ell)\right]$$

and also, formally:

$$\cos(y^\ell D)q(x) = \frac{1}{2}\left[q(x+iy^\ell) + q(x-iy^\ell)\right]$$

$$\sin(y^\ell D)q(x) = \frac{1}{2i}\left[q(x+iy^\ell) - q(x-iy^\ell)\right]$$

4 Convergence results

In this section we will recall an uniform estimate, with respect to j, for the convergence of series involving the H-KdF polynomials $H_n^{(j)}(x, y^\ell)$, for every real $\ell \geq 1$:

$$\sum_{n=0}^{\infty} a_n \, H_n^{(j)}(x, y^\ell). \tag{4.1}$$

Theorem 4.1 *For every real* $\ell \geq 1$, $j \geq 2$, $-\infty < x < +\infty$, $-\infty < y < +\infty$, $n = 0, 1, 2, \ldots,$ *the following estimate holds true:*

$$|H_n^{(j)}(x, y^\ell)| \leq n! \; exp\left\{|x| + |y^\ell|\right\} \;, \tag{4.2}$$

The proof is derived by the same method used in the book of D.V. Widder [20], p. 166, for the case when $j = 2$ and $\ell = 1$. A deep analysis of the convergence condition for the series (4.1) (in the above mentioned case), is performed in this book, however the relevant estimates can be only partially extended, since many of them are based on the integral representation of the H-KdF polynomials.

In general, assuming $j > 2$, an integral representation generalizing the Gauss-Weierstrass transform exists only when $j = 2q$ and q is an odd number, (see [13], [14]).

In the present lecture, we will limit ourselves to consider, as boundary (or initial) data, analytic functions $F(x) = \sum_{n=0}^{\infty} a_n x^n$ for which the coefficients a_n tend to zero sufficiently fast, in order to guarantee the convergence of the relevant series expansions.

To this aim, we only use the following theorem

Theorem 4.2 *Suppose there exists a number $\alpha > 1$, such that the coefficients a_n satisfy the following estimate:*

$$|a_n| = O\left(\frac{1}{n^\alpha n!}\right), \tag{4.3}$$

then, for every j, the series expansion (4.1) is absolutely and uniformly convergent in every bounded region of the (x, y) plane.

Proof. The result immediately follows from the estimate

$$\left|\sum_{n=0}^{\infty} a_n\ H_n^{(j)}(x,y)\right| \leq \sum_{n=0}^{\infty}\ |a_n|\left|H_n^{(j)}(x,y)\right| \leq e^{|x|+|y|}\sum_{n=0}^{\infty}\ |a_n|n!,$$

since that the last series is convergent by condition (4.3).

Remark 4.1 The condition (4.3) include analytic functions with polynomial growth at infinity, but not the exponential function e^x, whereas, in the case $j = 2$, the above mentioned book of Widder include all functions belonging to the so called Huygens class H^0, however, by the point of view of the Applied Analysis, the considered conditions are sufficient to cover all realistic situations, since it is reasonable to assume that all the boundary (or initial) data are expressed by bounded functions, vanishing at infinity, and negligible outside a suitable bounded interval.

5 First order systems, $(j = \ell = 1)$

Theorem 5.1 *Consider the first order system of partial differential equations, in the half plane* $y > 0$*:*

$$\begin{cases} \dfrac{\partial S_0}{\partial y} = \dfrac{\partial S_1}{\partial x} \\ \dfrac{\partial S_1}{\partial y} = \dfrac{\partial S_0}{\partial x} \\ S_0(x,0) = q_0(x) \\ S_1(x,0) = q_1(x), \end{cases} \tag{5.1}$$

where $q_0(x)$ *and* $q_1(x)$ *are given analytic functions.*

Then the operational solution of (5.1) is given by a pair of functions which are represented, in terms of definitions (3.1) and (3.5), as follows:

$$S_0(x,y) = \cosh(yD)\, q_0(x) + \sinh(yD)\, q_1(x) \ ,$$

$$S_1(x,y) = \sinh(yD)\, q_0(x) + \cosh(yD)\, q_1(x) \ .$$

Proof. A straightforward computation gives:

$$\frac{\partial S_0}{\partial y} = \sinh(yD)\ q_0'(x) + \cosh(yD)\ q_1'(x) = \frac{\partial S_1}{\partial x} \ ,$$

so that the first equation is satisfied.

On the other hand, we find:

$$\frac{\partial S_1}{\partial y} = \cosh(yD)\ q_0'(x) + \sinh(yD)\ q_1'(x) = \frac{\partial S_0}{\partial x} \ ,$$

and the boundary conditions, when $y = 0$, are trivially satisfied.

Remark 5.1 After differentiating both sides of both equations in system (5.1) we obtain the second order equation

$$\frac{\partial^2 S}{\partial y^2} = \frac{\partial^2 S}{\partial x^2}$$

which can be considered with two independent boundary conditions

$$S(x,0) = q_0(x), \qquad \text{and :} \quad \frac{\partial}{\partial y} S(x,0) = v_1(x), \tag{5.2}$$

(see e.g. [6]).

The solution of this problem is the sum of

$$S_0(x,y) := \cosh(yD)q_0(x) \qquad \text{and} \qquad S_1(x,y) := \sinh(yD)\int_0^x v_1(\xi)d\xi \ . \tag{5.3}$$

Remark 5.2 Note that a link between the boundary conditions of the system (5.1) and equation (5.2) is expressed by $v_1(x) = q_1'(x)$.

Consider now the classical Cauchy-Riemann system in the half-plane $y > 0$:

$$\begin{cases} \dfrac{\partial S_0}{\partial y} = -\dfrac{\partial S_1}{\partial x} \\ \dfrac{\partial S_1}{\partial y} = \dfrac{\partial S_0}{\partial x} \\ S_0(x,0) = q_0(x) \\ S_1(x,0) = q_1(x) \ . \end{cases} \tag{5.4}$$

In this case, the operational solution of (5.4) is given by a pair of functions which are represented, in terms of definitions (3.9) and (3.11), as follows:

$$S_0(x,y) = \cos(yD)\, q_0(x) - \sin(yD)\, q_1(x) \ , \tag{5.5}$$

$$S_1(x,y) = \sin(yD)\, q_0(x) + \cos(yD)\, q_1(x) \ . \tag{5.6}$$

The proof is obtained in the same way as before.

Remark 5.3 After differentiating both sides of both equations in system (5.4) respectively with respect y and x, we obtain the Laplace equation with Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial^2 S}{\partial y^2} = -\dfrac{\partial^2 S}{\partial x^2} \\ S(x,0) = q_0(x) \ , \end{cases} \tag{5.7}$$

and the operational solution of this problem is given by

$$S(x,y) := \cos(yD)q_0(x) \tag{5.8}$$

Note that, considering the Newmann problem in the half-plane $y > 0$:

$$\begin{cases} \dfrac{\partial^2 S}{\partial y^2} = -\dfrac{\partial^2 S}{\partial x^2} \\ \dfrac{\partial S}{\partial y} = q_1(x) \ , \end{cases} \tag{5.9}$$

then *one* formal solution is given by

$$S(x,y) := \sin(yD)\int_0^x q_1(\xi)d\xi \tag{5.10}$$

(obviously $D = D_x$).

As a matter of fact, we find:

$$\frac{\partial S}{\partial y} = \cos(yD)D\left(\int_0^x q_1(\xi)d\xi\right) = \cos(yD)q_1(x)\ ,$$

$$\frac{\partial^2 S}{\partial y^2} = -\sin(yD)Dq_1(x)\ ,$$

and

$$\frac{\partial S}{\partial x} = \sin(yD)D\left(\int_0^x q_1(\xi)d\xi\right) = \sin(yD)q_1(x)\ ,$$

$$\frac{\partial^2 S}{\partial x^2} = \sin(yD)Dq_1(x)\ ,$$

and the boundary condition, when $y = 0$, is trivially satisfied.

6 The case when $\ell = 1$ and $j = 2$

Consider the two systems of partial differential equations, in the half plane $y > 0$:

$$\begin{cases} \dfrac{\partial S_0}{\partial y} = \dfrac{\partial^2 S_1}{\partial x^2} \\ \dfrac{\partial S_1}{\partial y} = \dfrac{\partial^2 S_0}{\partial x^2} \\ S_0(x,0) = q_0(x) \\ S_1(x,0) = q_1(x), \end{cases} \tag{6.1}$$

or

$$\begin{cases} \dfrac{\partial S_0}{\partial y} = -\dfrac{\partial^2 S_1}{\partial x^2} \\ \dfrac{\partial S_1}{\partial y} = \dfrac{\partial^2 S_0}{\partial x^2} \\ S_0(x,0) = q_0(x) \\ S_1(x,0) = q_1(x), \end{cases} \tag{6.2}$$

where $q_0(x)$ and $q_1(x)$ are given analytic functions.

It is easily checked that

Theorem 6.1 *The system (6.1) admits the solution:*

$$\begin{aligned} S_0(x,y) &= \cosh\left(yD^2\right) q_0(x) + \sinh\left(yD^2\right) q_1(x) \ , \\ S_1(x,y) &= \sinh\left(yD^2\right) q_0(x) + \cosh\left(yD^2\right) q_1(x) \ . \end{aligned} \tag{6.3}$$

Analogously, the problem (6.2), admits the solution:

$$\begin{aligned} S_0(x,y) &= \cos\left(yD^2\right) q_0(x) - \sin\left(yD^2\right) q_1(x) \ , \\ S_1(x,y) &= \sin\left(yD^2\right) q_0(x) + \cos\left(yD^2\right) q_1(x) \ . \end{aligned} \tag{6.4}$$

The proof is obtained by exploiting the same technique we used in the preceding section.

By differentiating equations in system (6.1) we are lead to the problem (see [6], eq. (6.1)):

$$\begin{cases} \dfrac{\partial^2 S}{\partial y^2} = \dfrac{\partial^4 S}{\partial x^4}, \quad \text{in the half} - \text{plane} \quad y > 0, \\ S(x,0) = q(x), \qquad \dfrac{\partial}{\partial y} S(x,0) = v(x), \end{cases} \tag{6.5}$$

where $q(x) := q_0(x)$ and $v(x) := q_1''(x)$.

And analogously, from the system (6.2) we are lead to the problem:

$$\begin{cases} \dfrac{\partial^2 S}{\partial y^2} = -\dfrac{\partial^4 S}{\partial x^4}, \quad \text{in the half} - \text{plane} \quad y > 0, \\ S(x,0) = q(x), \qquad \dfrac{\partial}{\partial y} S(x,0) = v(x), \end{cases} \tag{6.6}$$

with two analytic boundary conditions $q(x)$, $v(x)$.

The above considerations yield the conclusion:

Theorem 6.2 *The problem (6.6) admits the solution:*

$$S(x,y) = \cos(yD^2) q(x) + \sin(yD^2) \int_0^x (x-\xi) v(\xi) d\xi \ .$$

7 The case when $\ell = 1$ and integer $j \geq 2$

In this case we can consider systems of the following type:

$$\begin{cases} \dfrac{\partial S_0}{\partial y} = \dfrac{\partial^j S_1}{\partial x^j} \\ \dfrac{\partial S_1}{\partial y} = \dfrac{\partial^j S_0}{\partial x^j} \\ S_0(x,0) = q_0(x) \\ S_1(x,0) = q_1(x), \end{cases} \tag{7.1}$$

or

$$\begin{cases} \dfrac{\partial S_0}{\partial y} = -\dfrac{\partial^j S_1}{\partial x^j} \\ \dfrac{\partial S_1}{\partial y} = \dfrac{\partial^j S_0}{\partial x^j} \\ S_0(x,0) = q_0(x) \\ S_1(x,0) = q_1(x). \end{cases} \tag{7.2}$$

In this case, we have the result:

Theorem 7.1 *The system (7.1) admits the solution:*

$$\begin{aligned} S_0(x,y) &= \cosh\left(yD^j\right) q_0(x) + \sinh\left(yD^j\right) q_1(x) \ , \\ S_1(x,y) &= \sinh\left(yD^j\right) q_0(x) + \cosh\left(yD^j\right) q_1(x). \end{aligned} \tag{7.3}$$

Analogously, the problem (7.2), admits the solution:

$$\begin{aligned} S_0(x,y) &= \cos\left(yD^j\right) q_0(x) - \sin\left(yD^j\right) q_1(x) \ , \\ S_1(x,y) &= \sin\left(yD^j\right) q_0(x) + \cos\left(yD^j\right) q_1(x). \end{aligned} \tag{7.4}$$

After differentiating the first of equations (7.1), we are lead to the problem:

$$\begin{cases} \dfrac{\partial^2 S}{\partial y^2} = \dfrac{\partial^{2j} S}{\partial x^{2j}}, \quad \text{in the half} - \text{plane} \quad y > 0, \\ S(x,0) = q(x), \qquad \dfrac{\partial}{\partial y} S(x,0) = v(x), \end{cases} \tag{7.5}$$

where $q(x) := q_0(x)$ and $v(x) := q_1^{(j)}(x)$.

And analogously, from the system (7.2) we are lead to the problem:

$$\begin{cases} \dfrac{\partial^2 S}{\partial y^2} = -\dfrac{\partial^{2j} S}{\partial x^{2j}}, \quad \text{in the half} - \text{plane} \quad y > 0, \\ \\ S(x,0) = q(x), \qquad \dfrac{\partial}{\partial y} S(x,0) = v(x) \ . \end{cases} \tag{7.6}$$

Therefore we can proclaim the result:

Theorem 7.2 *The problem (7.5) admits the solution:*

$$S(x,y) = \cosh(yD^j)q(x) + \sinh(yD^j) \int_0^x \frac{(x-\xi)^{j-1}}{(j-1)!} v(\xi) d\xi,$$

and problem (7.6) the following one:

$$S(x,y) = \cos(yD^j)q(x) + \sin(yD^j) \int_0^x \frac{(x-\xi)^{j-1}}{(j-1)!} v(\xi) d\xi,$$

Remark 7.1 Note that similar results can be obtained by considering in systems studied in the above sections boundary conditions of the following forms:

$$\begin{cases} S_0(x,0) = q_0(x) \\ \dfrac{\partial}{\partial y} S_0(x,0) = w_0(x), \end{cases}$$

or

$$\begin{cases} \dfrac{\partial}{\partial y} S_0(x,0) = w_0(x) \\ \dfrac{\partial}{\partial y} S_1(x,0) = w_1(x). \end{cases}$$

8 The case for any real $\ell \geq 1$ and $j = 1$

Note that equations in system extending (5.1), assuming $\ell > 1$, degenerate on the boundary when $y \to 0$. For example, for $\ell > 1$ and $j = 1$, we have to consider the system:

$$\begin{cases} \dfrac{\partial S_0}{\partial y} = \ell y^{\ell-1} \dfrac{\partial S_1}{\partial x} \\ \dfrac{\partial S_1}{\partial y} = \ell y^{\ell-1} \dfrac{\partial S_0}{\partial x} \\ S_0(x,0) = q_0(x) \\ S_1(x,0) = q_1(x), \end{cases} \tag{8.1}$$

where $q_0(x)$ and $q_1(x)$ are given analytic functions.

Using the same methods as before, we find the result:

Theorem 8.1 *The system (8.1) admits the solution:*

$$S_0(x,y) = \cosh(y^\ell D)q_0(x) + \sinh(y^\ell D)q_1(x) \ ,$$
$$S_1(x,y) = \sinh(y^\ell D)q_0(x) + \cosh(y^\ell D)q_1(x) \ .$$

Differentiating the first equation in system (8.1) we find:

$$\frac{\partial^2 S_0}{\partial y^2} = \ell(\ell-1)y^{\ell-2}\frac{\partial S_1}{\partial x} + \ell y^{\ell-1}\frac{\partial}{\partial x}\frac{\partial S_1}{\partial y} \ .$$

and furthermore

$$\frac{\partial^2 S_0}{\partial y^2} = \frac{\ell(\ell-1)y^{\ell-2}}{\ell y^{\ell-1}}\frac{\partial S_0}{\partial y} + \ell y^{\ell-1}\frac{\partial}{\partial x}\left(\ell y^{\ell-1}\frac{\partial S_0}{\partial x}\right) \ ,$$

so that we are lead to the problem:

$$\begin{cases} \dfrac{\partial^2 S}{\partial y^2} = \dfrac{(\ell-1)}{y}\dfrac{\partial S}{\partial y} + \ell^2 y^{2(\ell-1)}\dfrac{\partial^2 S}{\partial x^2} \qquad \text{in the half}-\text{plane} \quad y>0, \\ S(x,0) = q_0(x), \end{cases} \tag{8.2}$$

with analytic boundary condition $q_0(x)$.

Remark 8.1 Note that in this case it is not possible to prescribe arbitrarily a second condition since, being $\ell > 1$, it must be necessarily: $\frac{\partial}{\partial y}S(x,0) = 0$.

Therefore we can proclaim the result:

Theorem 8.2 *The problem (8.2) admits the solution:*

$$S(x,y) = \cosh(y^\ell D)q_0(x) \ .$$

8.1 A generalized Cauchy-Riemann system

Considering again the case when $\ell > 1$ and $j = 1$, the system:

$$\begin{cases} \dfrac{\partial S_0}{\partial y} = -\ell y^{\ell-1}\dfrac{\partial S_1}{\partial x} \\ \dfrac{\partial S_1}{\partial y} = \ell y^{\ell-1}\dfrac{\partial S_0}{\partial x} \\ S_0(x,0) = q_0(x) \\ S_1(x,0) = q_1(x), \end{cases} \tag{8.3}$$

(where $q_0(x)$ and $q_1(x)$ are given analytic functions), can be considered as a generalization of the classical Cauchy-Riemann system.

In this case, we find the result:

Theorem 8.3 *The system (8.3) admits the solution:*

$$S_0(x,y) = \cos(y^\ell D)q_0(x) - \sin(y^\ell D)q_1(x) \ ,$$
$$S_1(x,y) = \sin(y^\ell D)q_0(x) + \cos(y^\ell D)q_1(x) \ .$$

The corresponding second order problem has the form:

$$\begin{cases} \dfrac{\partial^2 S}{\partial y^2} = \dfrac{(\ell-1)}{y}\dfrac{\partial S}{\partial y} - \ell^2 y^{2(\ell-1)}\dfrac{\partial^2 S}{\partial x^2} \qquad \text{in the half} - \text{plane} \quad y>0, \\ S(x,0) = q_0(x) \ . \end{cases} \tag{8.4}$$

Therefore we can proclaim the result:

Theorem 8.4 *The problem (8.4) admits the solution:*

$$S(x,y) = \cos(y^\ell D)q_0(x) \ .$$

9 The general case: for any real $\ell > 1$ and integer $j \geq 2$

In the half-plane $y > 0$, we consider the problem:

$$\begin{cases} \dfrac{\partial S_0}{\partial y} = \ell y^{\ell-1}\dfrac{\partial^j S_1}{\partial x^j} \\ \dfrac{\partial S_1}{\partial y} = \ell y^{\ell-1}\dfrac{\partial^j S_0}{\partial x^j} \\ S_0(x,0) = q_0(x) \\ S_1(x,0) = q_1(x), \end{cases} \tag{9.1}$$

with analytic boundary conditions $q_0(x)$ and $q_1(x)$, or the problem

$$\begin{cases} \dfrac{\partial^2 S}{\partial y^2} = \dfrac{(\ell-1)}{y}\dfrac{\partial S}{\partial y} + \ell^2 y^{2(\ell-1)}\dfrac{\partial^{2j} S}{\partial x^{2j}} \\ S(x,0) = q_0(x). \end{cases} \tag{9.2}$$

Note that even in this case, it must be $\dfrac{\partial S}{\partial y}(x,0) = 0.$

Then

Theorem 9.1 *The problem (9.1) admits the solution:*

$$S_0(x,y) = \cosh(y^\ell D^j)q_0(x) + \sinh(y^\ell D^j)q_1(x),$$
$$S_1(x,y) = \sinh(y^\ell D^j)q_0(x) + \cosh(y^\ell D^j)q_1(x)$$

and the problem (9.2) the following one:

$$S(x,y) = \cosh(y^\ell D^j)q_0(x).$$

Analogously, considering the problems:

$$\begin{cases} \dfrac{\partial S_0}{\partial y} = -\ell y^{\ell-1}\dfrac{\partial^j S_1}{\partial x^j} \\ \dfrac{\partial S_1}{\partial y} = \ell y^{\ell-1}\dfrac{\partial^j S_0}{\partial x^j} \\ S_0(x,0) = q_0(x) \\ S_1(x,0) = q_1(x), \end{cases} \tag{9.3}$$

with analytic boundary conditions $q_0(x)$ and $q_1(x)$, or

$$\begin{cases} \dfrac{\partial^2 S}{\partial y^2} = \dfrac{(\ell-1)}{y}\dfrac{\partial S}{\partial y} - \ell^2 y^{2(\ell-1)}\dfrac{\partial^{2j} S}{\partial x^{2j}} \\ S(x,0) = q_0(x). \end{cases} \tag{9.4}$$

we have the result:

Theorem 9.2 *The problem (9.3) admits the solution:*

$$S_0(x,y) = \cos(y^\ell D^j)q_0(x) - \sin(y^\ell D^j)q_0(x)\ ,$$
$$S_1(x,y) = \sin(y^\ell D^j)q_0(x) + \cos(y^\ell D^j)q_0(x)$$

and the problem (9.4) the following one:

$$S(x,y) = \cos(y^\ell D^j)q_0(x).$$

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Meromorphic functions sharing small functions with their derivatives

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Abstract: In this paper, we study the problem of uniqueness of meromorphic functions sharing small functions with their derivatives, partly answer the conjecture of R. Brüch, and obtain some results which improve the theorems of K. Yu et al.

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Keywords: meromorphic function; sharing value; uniqueness.

1. Introduction and Main Results

In this paper, a meromorphic function means meromorphic in the open complex plane. We use the usual notations of Nevanlinna theory of meromorphic functions as explained in [1,7]. E (respectively, I) will be denoted by a set of finite (respectively, infinite) linear measure, not necessary the same at each occurrence. For a complex number a, we say that f and g share the value a CM (respectively, IM) provided that f and g have the same $a-$points counting multiplicities (respectively, ignoring multiplicities).

On the uniqueness problem of entire functions or meromorphic functions with their first or $k-$th derivatives involving two CM or IM values has been studied by many mathematicians (see[1]). However, these results almost concerned two shared values. As for entire functions or meromorphic functions with their first or $k-$th derivative sharing only one value or only one small function CM, R. Brüch posed the following conjecture in [6]:

Conjecture 1: *Let f be a nonconstant entire function. Suppose that $\rho_1(f) < \infty$, $\rho_1(f)$ is not a positive integer, f and f' share one finite value a CM, then*

$$\frac{f'-a}{f-a} = c$$

for some nonzero constant c, where $\rho_1(f)$ is the first iterated order of f, that is

$$\rho_1(f) = \limsup_{r\to\infty} \frac{log^+log^+T(r,f)}{logr}.$$

In [6], R. Brüch also proved that the conjecture 1 holds if $a = 0$ and $N(r, \frac{1}{f'}) = S(r, f)$.

Example 2. Let $f = \frac{2e^z+z+1}{e^z+1}$, it is easy to see that f and f' share 1 CM, and $\frac{f'-1}{f-1} = c$ does not hold for some nonzero constant c.

The Example 2 shows that the conjecture 1 is invalid for meromorphic functions.

Example 3. Let $f = e^{e^z} + e^z$, it is also easy to see f and f' share small function e^z CM, but $f \not\equiv f'$.

The Example 3 shows that the conjecture 1 is invalid for entire functions with infinity order.

Since then, a considerable results have been gotten in this or its related topics (see [2],[3],[5] etc.), in order to solve the conjecture 1. In [2], K. W. Yu proved the following theorem:

Theorem 4. Let f be a nonconstant entire function and a be a meromorphic function such that $a \not\equiv 0, \infty$ and $T(r,a) = o(T(r,f))$ as $r \to \infty$. If $f - a$, $f^{(k)} - a$ share the value 0 CM and $\delta(0,f) > \frac{3}{4}$, then $f \equiv f^{(k)}$, where k is a positive integer.

Theorem 5. Let f be a nonconstant meromorphic function and a be a meromorphic function such that $a \not\equiv 0, \infty$, f and a have no any common pole and $T(r,a) = o(T(r,f))$ as $r \to \infty$. If $f - a$, $f^{(k)} - a$ share the value 0 CM and

$$4\delta(0,f) + 2(8+k)\Theta(\infty,f) > 19 + 2k, \tag{1}$$

then $f \equiv f^{(k)}$, where k is a positive integer.

Theorem 6. Suppose that k is an odd integer. Then Theorem 5 is valid for all small function a.

In [2], K. W. Yu posed the following questions:

Question 7. *Is the condition $\delta(0,f) > \frac{3}{4}$ sharp in Theorem 4?*

Question 8. *Can the condition " f and a have no any common pole " be deleted in Theorem 5?*

The purpose of this paper is to solve the above questions. We prove the following results which are improvements of Theorem 4, Theorem 5 and Theorem 6.

Theorem 9. Let f be a nonconstant meromorphic function and a be a meromorphic function such that $a \not\equiv 0, \infty$, and $T(r,a) = o(T(r,f))$ as $r \to \infty$. If $f - a$, $f^{(k)} - a$ share the value 0 CM and

$$2\delta(0,f) + (4+k)\Theta(\infty,f) > 5 + k, \tag{2}$$

where k is a positive integer, then $f \equiv f^{(k)}$.

From Theorem 9 we have

Corollary 10. Let f be a nonconstant entire function and a be a meromorphic function such that $a \not\equiv 0, \infty$, and $T(r,a) = o(T(r,f))$ as $r \to \infty$. If $f - a$, $f^{(k)} - a$ share the value 0 CM and $\delta(0,f) > \frac{1}{2}$, where k is a positive integer, then $f \equiv f^{(k)}$.

Clearly, Corollary 10 is an improvement of Theorem 4. Theorem 9 is an improvement of theorem 5.

2. Some Lemmas

In order to prove our result, the following lemmas will be used.

Lemma 11([1]). Let f be a nonconstant meromorphic function and k be a positive integer, then

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k\overline{N}(r,f) + S(r,f).$$

Lemma 12([4]). Let $f_1,\ f_2,\ f_3$ be nonconstant meromorphic functions such that $f_1+f_2+f_3\equiv 1$. If $f_1,\ f_2,\ f_3$ are linearly independent, then

$$T(r,f_1)<\sum_{j=1}^{3}N(r,\frac{1}{f_j})+\sum_{j=1}^{3}\overline{N}(r,f_j)+o(T(r)),$$

where $T(r)=\max_{1\leq j\leq 3}\{T(r,f_j)\}\ (r\notin E)$.

From the second fundamental theorem, we can obtain

Lemma 13. Let $f_1,\ f_2$ be nonconstant meromorphic functions and $c_1,\ c_2,\ c_3$ be nonzero constants. If $c_1f_1+c_2f_2\equiv c_3$, then

$$T(r,f_1)<\overline{N}(r,\frac{1}{f_1})+\overline{N}(r,\frac{1}{f_2})+\overline{N}(r,f_1)+S(r,f_1).$$

3. Proof of Theorem 9

In order to prove Theorem 9, we set

$$F=\frac{f^{(k)}-a}{f-a}. \tag{3}$$

Since $f-a,\ f^{(k)}-a$ share the value 0 CM, so F has the form he^{α}, where α is an entire function and h is a meromorphic function such that

$$\overline{N}(r,h)\leq\overline{N}(r,f)+S(r,f),\quad N(r,\frac{1}{h})=S(r,f). \tag{4}$$

Clearly, from (3), we have

$$f_1+f_2+f_3=1, \tag{5}$$

where $f_1=\frac{f^{(k)}}{a}$, $f_2=-\frac{fhe^{\alpha}}{a}$ and $f_3=he^{\alpha}$.

By (2) and the definition of $\delta(0,f)$, we find $m(r,\frac{1}{f})\geq(\delta(0,f)-\varepsilon)T(r,f)$ for any given $\varepsilon\,(0<\varepsilon<\delta(0,f))$, and hence

$$(\delta(0,f)-\varepsilon)T(r,f)\leq T(r,f^{(k)})+S(r,f)\leq(k+1)T(r,f)+S(r,f). \tag{6}$$

From (6), we know that f_1 is nonconstant. Suppose that $f_2=c$, where c is a constant. If $c=1$, then it follows from (5) that

$$ff^{(k)}\equiv a^2. \tag{7}$$

It implies by the fundamental estimate of logarithmic derivative and (7) that

$$2T(r,f)=m(r,\frac{f^{(k)}}{f})+N(r,\frac{f^{(k)}}{f})+S(r,f)=S(r,f),$$

which is impossible, so $c\neq 1$. By (5), we get

$$\frac{f^{(k)}}{a}+he^{\alpha}=1-c, \tag{8}$$

which leads by (8), (4), Lemma 13 and Lemma 11 that

$$\begin{aligned}
T(r,f) &= T(r,f^{(k)})+S(r,f)\\
&\leq \overline{N}(r,\frac{1}{f^{(k)}})+\overline{N}(r,f^{(k)})+S(r,f)\\
&\leq N(r,\frac{1}{f})+(k+1)\overline{N}(r,f)+S(r,f)\\
&\leq 2N(r,\frac{1}{f})+(k+4)\overline{N}(r,f)+S(r,f),
\end{aligned}$$

which contradicts with (2). Suppose that $f_3=c$, where c is a constant. If $c\neq 1$, by (5), we get

$$\frac{f^{(k)}}{f}-c=\frac{a(1-c)}{f},$$

it means from the fundamental estimate of logarithmic derivative that $m(r,\frac{1}{f})=S(r,f)$, and so $\delta(0,f)=0$, which also contradicts with (2). Therefore, $c=1$ and Theorem 9 follows.

Hence in the following we may assume that f_1, f_2, f_3 are nonconstant. We consider two cases.

Suppose that f_1, f_2, f_3 are linearly independent. By Lemma 12, Lemma 11 and (4),we deduce

$$\begin{aligned}
T(r,f_1) &\leq N(r,\frac{1}{f^{(k)}})+N(r,\frac{1}{f})+2\overline{N}(r,f)+2\overline{N}(r,h)+S(r,f)\\
&\leq 2N(r,\frac{1}{f})+(k+4)\overline{N}(r,f)+S(r,f),
\end{aligned}$$

$$\begin{aligned}
T(r,f_2) &\leq N(r,\frac{1}{f^{(k)}})+N(r,\frac{1}{f})+2\overline{N}(r,f)+2\overline{N}(r,h)+S(r,f)\\
&\leq 2N(r,\frac{1}{f})+(k+4)\overline{N}(r,f)+S(r,f),
\end{aligned}$$

and

$$\begin{aligned}
T(r,f_3) &\leq N(r,\frac{1}{f^{(k)}})+N(r,\frac{1}{f})+2\overline{N}(r,h)+2\overline{N}(r,f)+S(r,f)\\
&\leq 2N(r,\frac{1}{f})+(k+4)\overline{N}(r,f)+S(r,f).
\end{aligned}$$

Combining above three inequalities, we have

$$\begin{aligned}
T(r) &\leq 2N(r,\frac{1}{f})+(k+4)\overline{N}(r,f)+o(T(r))\\
&\leq (2(1-\delta(0,f))+(k+4)(1-\Theta(\infty,f)))T(r)+o(T(r)),
\end{aligned}$$

it follows from (2) that $T(r)=o(T(r))$, where $T(r)$ is defined in Lemma 12, which is also impossible.

Suppose that f_1, f_2, f_3 are linearly dependent. Then, there exist three constants $(c_1, c_2, c_3) \neq (0,0,0)$ such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0. \tag{9}$$

If $c_1 = 0$, it means by (9) that $f = \frac{ac_3}{c_2}$, and so $T(r,f) = S(r,f)$, which is impossible. Thus, $c_1 \neq 0$ and

$$f_1 = -\frac{c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3. \tag{10}$$

Now combining (5) and (10), we get

$$(1 - \frac{c_2}{c_1}) f_2 + (1 - \frac{c_3}{c_1}) f_3 = 1,$$

i.e

$$-(1 - \frac{c_2}{c_1}) \frac{he^{\alpha}}{a} f + (1 - \frac{c_3}{c_1}) he^{\alpha} = 1. \tag{11}$$

Again by (11), (4), the first main theorem and Lemma 13, we have

$$\begin{aligned} T(r,f) &= T(r, he^{\alpha}) + S(r,f) \\ &\leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r,h) + S(r,f) \\ &\leq 2N(r, \frac{1}{f}) + (k+4)\overline{N}(r,f) + S(r,f), \end{aligned}$$

which also contradicts with (2).

This proves Theorem 9.

4. Final Remarks

Remark 14. The Example 2 and Example 3 show that (2) is necessary.

Remark 15. If we compare our results with the conjecture 1 of R. Brüch, it is easy to see that we do not assume any restriction on the growth of f.

Remark 16. By (1), we know that $\delta(0,f) > \frac{3}{4}$ and $\Theta(\infty, f) > \frac{15+2k}{16+2k}$. However, By (2), we know that $\delta(0,f) > \frac{1}{2}$ and $\Theta(\infty, f) > \frac{3+k}{4+k}$. It is easy to see that (2) is better than (1).

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Counterparts of Schwarz's inequality for Čebyšev functional

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Abstract: A counterpart of Schwarz's inequality for Čebyšev functional and some determinantal inequalities are proved.

Key words and phrases: Schwarz's Inequality, Čebyšev functional, Determinantal inequality.

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1 Introduction

Recently, S. S. Dragomir [1] proved a new counterpart of Schwarz's inequality in inner product spaces. In real discrete case his results can be stated as:

Theorem 1 *Let $a = (a_1, \ldots, a_n)$, $a = (b_1, \ldots, b_n)$ and $p = (p_1, \ldots, p_n)$ be n-tuples such that*

$$0 < m \le \frac{a_i}{b_i} \le M < \infty \qquad (i = 1, \ldots, n). \tag{1}$$

Then

$$0 \leq \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 - \left(\sum_{i=1}^{n} p_i a_i b_i \right)^2 \leq \frac{1}{4} (M-m)^2 \left(\sum_{i=1}^{n} p_i b_i^2 \right)^2 . \tag{2}$$

In, fact S. S. Dragomir give a weaker condition then (1) which will not be stated here.
This result is of special interest since we can use it in the proof of well-known Grüss inequality.
Namely, denote by $T_n(a,b;p)$ the Čebyšev's functional:

$$T_n(a,b;p) = \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i, \tag{3}$$

where $P_n = \sum_{i=1}^{n} p_i$. It is well known that the following inequality of Cauchy type is valid

$$(T_n(a,b;p))^2 \leq T_n(a,a;p) T_n(b,b;p). \tag{4}$$

if a and b satisfy the condition

$$m_1 \leq a_i \leq M_1, \quad m_2 \leq b_i \leq M_2 \qquad (i = 1, \ldots, n), \tag{5}$$

we have from (2) for $b_i = 1 \quad (i = 1, \ldots, n)$:

$$T_n(a,a;p) \leq \frac{1}{4} (M_1 - m_1)^2 .$$

Therefore (4) gives the well-known Grüss inequality

$$|T_n(a,b;p)| \leq \frac{1}{4} (M_1 - m_1)(M_2 - m_2) . \tag{6}$$

In this paper we give related results involving Čebyšev's functional $T_n(a,b;p)$ as well as some of its extensions.

2 Results for Čebyšev's Functional

Theorem 2 *Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ and $p = (p_1, \ldots, p_n)$ be n-tuples such that for any $i, j \in \{1, \ldots, n\}$ with $i < j$*

$$m (b_j - b_i) \leq a_j - a_i \leq M (b_j - b_i) , \tag{7}$$

where m, M are given positive real numbers. Then we have inequality

$$0 \le T_n(a,a;p)T_n(b,b;p) - (T_n(a,b;p))^2 \le \frac{1}{4}(M-m)^2(T_n(b,b;p))^2. \quad (8)$$

Proof. The first inequality in (8) is (4). Let us note that S. S. Dragomir has proved in [2] that the following result is valid if the assumption of Theorem 2 is fulfilled:

$$(m+M)\,T_n(a,b;p) \ge T_n(a,a;p) + mMT_n(b,b;p). \quad (9)$$

Note that $T_n(b,b;p) \ge 0$. So multiplying (9) by $T_n(b,b;p)$ we have

$$T_n(a,a;p)T_n(b,b;p) + mM\,(T_n(b,b;p))^2 \le (m+M)\,T_n(a,b;p)T_n(b,b;p).$$

Therefore

$$\begin{aligned}
&T_n(a,a;p)T_n(b,b;p) - (T_n(a,b;p))^2 \\
&\le (m+M)\,T_n(a,b;p)T_n(b,b;p) - mM\,(T_n(b,b;p))^2 - (T_n(a,b;p))^2 \\
&= \frac{1}{4}(M-m)^2(T_n(b,b;p))^2 - \left(\frac{m+M}{2}T_n(b,b;p) - T_n(a,b;p)\right)^2 \\
&\le \frac{1}{4}(M-m)^2(T_n(b,b;p))^2.
\end{aligned}$$

□

Theorem 3 *Let $f,g:[\alpha,\beta]\to\mathbf{R}$ be continuous on $[\alpha,\beta]$ and differentiable on (a,b) with $g'(x)\ne 0$ for $x\in(\alpha,\beta)$. Assume also that*

$$-\infty < \gamma = \inf_{x\in(\alpha,\beta)} \frac{f'(x)}{g'(x)}, \qquad \sup_{x\in(\alpha,\beta)} \frac{f'(x)}{g'(x)} = \Gamma < \infty. \quad (10)$$

If $\bar{x} = (x_1,\ldots,x_n)$ is a real n-tuple with $x_i\in[\alpha,\beta]$ and $x_i\ne x_j$ for $i\ne j$ and if we denote by $f(\bar{x})$ the n-tuple $(f(x_1),\ldots,f(x_n))$, then we have the inequality

$$\begin{aligned}
0 &\le T_n(f(\bar{x}),f(\bar{x});p)\,T_n(g(\bar{x}),g(\bar{x});p) - (T_n(f(\bar{x}),g(\bar{x});p))^2 \\
&\le \frac{1}{4}(\Gamma-\gamma)^2(T_n(g(\bar{x}),g(\bar{x});p))^2. \quad (11)
\end{aligned}$$

Proof. Applying the Cauchy Mean-value Theorem, for any $i, j \in \{1, \ldots, n\}$ with $i < j$ there exist $\xi_{ij} \in (\alpha, \beta)$ so that

$$\frac{f(x_j) - f(x_i)}{g(x_j) - f(x_i)} = \frac{f'(\xi_{ij})}{g'(\xi_{ij})} \in [\gamma, \Gamma].$$

S. S. Dragomir has proved in [2] that the following result is valid if the assumption of Theorem 3 is fulfilled:

$$(\gamma + \Gamma) T_n(f(\bar{x}), g(\bar{x}); p) \geq T_n(f(\bar{x}), f(\bar{x}); p) + \gamma\Gamma T_n(g(\bar{x}), g(\bar{x}); p). \quad (12)$$

for any $p = (p_1, \ldots, p_n)$ with $p_i \geq 0$, $i \in \{1, \ldots, n\}$. Note that $T_n(g(\bar{x}), g(\bar{x}); p) \geq 0$. So multiplying (12) by $T_n(g(\bar{x}), g(\bar{x}); p)$ we have

$$\begin{aligned} & T_n(f(\bar{x}), f(\bar{x}); p) T_n(g(\bar{x}), g(\bar{x}); p) + \gamma\Gamma (T_n(g(\bar{x}), g(\bar{x}); p))^2 \\ & \leq (\gamma + \Gamma) T_n(f(\bar{x}), g(\bar{x}); p) T_n(g(\bar{x}), g(\bar{x}); p). \end{aligned}$$

Therefore

$$\begin{aligned} & T_n(f(\bar{x}), f(\bar{x}); p) T_n(g(\bar{x}), g(\bar{x}); p) - (T_n(g(\bar{x}), g(\bar{x}); p))^2 \\ & \leq (\gamma + \Gamma) T_n(f(\bar{x}), g(\bar{x}); p) T_n(g(\bar{x}), g(\bar{x}); p) - \gamma\Gamma (T_n(g(\bar{x}), g(\bar{x}); p))^2 \\ & \quad - (T_n(f(\bar{x}), g(\bar{x}); p))^2 \\ & = \frac{1}{4}(\Gamma - \gamma)^2 (T_n(g(\bar{x}), g(\bar{x}); p))^2 \\ & \quad - \left(\frac{\gamma + \Gamma}{2} T_n(g(\bar{x}), g(\bar{x}); p) - T_n(f(\bar{x}), g(\bar{x}); p)\right)^2 \\ & \leq \frac{1}{4}(\Gamma - \gamma)^2 (T_n(g(\bar{x}), g(\bar{x}); p))^2. \end{aligned}$$

□

Theorem 4 *Assume that $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ are real n-tuples, $p = (p_1, \ldots, p_n)$ is a positive n-tuples and $b_i \neq A_n(a, p) = \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i$ for each $i \in \{1, \ldots, n\}$ where $P_n = \sum_{i=1}^{n} p_i$. If*

$$-\infty < l \leq \frac{a_i - A_n(a, p)}{b_i - A_n(b, p)} \leq L < \infty, \quad (13)$$

for any $i \in \{1, \ldots, n\}$, where l, L are given real numbers, then one has the inequality

$$0 \leq T_n(a, a; p) T_n(b, b; p) - (T_n(a, b; p))^2 \leq \frac{1}{4}(L - l)^2 (T_n(b, b; p))^2. \quad (14)$$

Proof. The first inequality in (14) is (4). Let us note that S. S. Dragomir has proved in [2] that the following result is valid if the hypothesis of Theorem 4 is fulfilled:

$$(l+L)\,T_n\,(a,b;p) \geq T_n\,(a,a;p) + lLT_n\,(b,b;p)\,. \tag{15}$$

Note that $T_n\,(b,b;p) \geq 0$. So multiplying (15) by $T_n\,(b,b;p)$ we have

$$T_n\,(a,a;p)\,T_n\,(b,b;p) + lL\,(T_n\,(b,b;p))^2 \leq (l+L)\,T_n\,(a,b;p)\,T_n\,(b,b;p)\,.$$

Therefore

$$\begin{aligned}
&T_n(a,a;p)T_n(b,b;p) - (T_n(a,b;p))^2 \\
&\leq (l+L)\,T_n(a,b;p)T_n(b,b;p) - lL\,(T_n(b,b;p))^2 - (T_n(a,b;p))^2 \\
&= \frac{1}{4}\,(L-l)^2\,(T_n(b,b;p))^2 - \left(\frac{l+L}{2}T_n(b,b;p) - T_n(a,b;p)\right)^2 \\
&\leq \frac{1}{4}\,(L-l)^2\,(T_n(b,b;p))^2\,.
\end{aligned}$$

□

3 Related Determinantal Inequalities

B. Mond, J. Pečarić and B. Tepeš [3] used the following matrix notation

$$\bar{x} = (x_1,\ldots,x_n)\,, \quad \bar{f}\,(\bar{x}) = [f_i\,(x_j)] = \begin{bmatrix} f_1\,(x_1) & \ldots & f_1\,(x_n) \\ \ldots & \ldots & \ldots \\ f_n\,(x_1) & \ldots & f_n\,(x_n) \end{bmatrix},$$

$$\det \bar{f}\,(\bar{x}) = \begin{vmatrix} f_1\,(x_1) & \ldots & f_1\,(x_n) \\ \ldots & \ldots & \ldots \\ f_n\,(x_1) & \ldots & f_n\,(x_n) \end{vmatrix}, \quad p\,(\bar{x}) = p\,(x_1)\cdot\ldots\cdot p\,(x_n)\,.$$

to prove results incorporated in

Theorem 5 *Let $f_1,\ldots,f_n$ and $g_1,\ldots,g_n$ with $p\,(x)>0$ be continuous functions. Assume also that $\det \bar{f}\,(\bar{x})$ and $\det \bar{g}\,(\bar{x})$ for $a \leq x_1 \leq \cdots \leq x_n \leq b$ satisfy the inequality*

$$m \det \bar{g}\,(\bar{x}) \leq \det \bar{f}\,(\bar{x}) \leq M \det \bar{g}\,(\bar{x})\,, \tag{16}$$

where m, M are given positive real numbers. Then we have the inequality

$$\begin{aligned}
0 &\leq \det\left[\int_a^b p(x)\, f_i(x)\, f_j(x)\, dx\right] \cdot \det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right] \\
&\quad - \left(\det\left[\int_a^b p(x)\, f_i(x)\, g_j(x)\, dx\right]\right)^2 \qquad (17)\\
&\leq \frac{1}{4}(M-m)^2 \left(\det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right]\right)^2.
\end{aligned}$$

Proof. The first inequality in (17) is the integral version of Cauchy's inequality [4, p. 600]. Let us note that B. Mond, J. Pečarić and B. Tepeš have proved in [3] that the following result is valid if the assumption of Theorem 5 is fulfilled:

$$\begin{aligned}
&(m+M)\det\left[\int_a^b p(x)\, f_i(x)\, g_j(x)\, dx\right] \\
&\geq \det\left[\int_a^b p(x)\, f_i(x)\, f_j(x)\, dx\right] + mM\det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right]. \qquad (18)
\end{aligned}$$

Note that $\det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right] \geq 0$. So multiplying (18) by $\det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right]$ we have

$$\begin{aligned}
&\det\left[\int_a^b p(x)\, f_i(x)\, f_j(x)\, dx\right] \det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right] \\
&\quad + mM\left(\det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right]\right)^2 \\
&\leq (m+M)\det\left[\int_a^b p(x)\, f_i(x)\, g_j(x)\, dx\right] \det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\det\left[\int_a^b p(x)\, f_i(x)\, f_j(x)\, dx\right] \det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right] \\
&\quad - \left(\det\left[\int_a^b p(x)\, f_i(x)\, g_j(x)\, dx\right]\right)^2
\end{aligned}$$

$$
\begin{aligned}
&\leq (m+M)\det\left[\int_a^b p(x)\,f_i(x)\,g_j(x)\,dx\right]\det\left[\int_a^b p(x)\,g_i(x)\,g_j(x)\,dx\right]\\
&\quad -mM\left(\det\left[\int_a^b p(x)\,g_i(x)\,g_j(x)\,dx\right]\right)^2-\left(\det\left[\int_a^b p(x)\,f_i(x)\,g_j(x)\,dx\right]\right)^2\\
&=\frac{1}{4}(M-m)^2\left(\det\left[\int_a^b p(x)\,g_i(x)\,g_j(x)\,dx\right]\right)^2\\
&\quad -\left(\frac{m+M}{2}\det\left[\int_a^b p(x)\,g_i(x)\,g_j(x)\,dx\right]\right)^2-\left(\det\left[\int_a^b p(x)\,f_i(x)\,g_j(x)\,dx\right]\right)^2\\
&\leq\frac{1}{4}(M-m)^2\left(\det\left[\int_a^b p(x)\,g_i(x)\,g_j(x)\,dx\right]\right)^2.
\end{aligned}
$$

□

Theorem 6 *Let $f_1,\dots,f_n$ and $g_1,\dots,g_n$ with $p(x)>0$ be continuous functions on $a\leq x_1\leq\cdots\leq x_n\leq b$ with $\det\left[g_i^{(j-1)}(x)\right]\neq 0$ for $x\in(a,b)$. Assume that*

$$
-\infty<\gamma=\inf_{x\in(a,b)}\frac{\det\left[f_i^{(j-1)}\right]}{\det\left[f_i^{(j-1)}\right]},\qquad \sup_{x\in(a,b)}\frac{\det\left[f_i^{(j-1)}\right]}{\det\left[f_i^{(j-1)}\right]}=\Gamma<\infty. \tag{19}
$$

Then we have the inequality

$$
\begin{aligned}
0&\leq\det\left[\int_a^b p(x)\,f_i(x)\,f_j(x)\,dx\right]\cdot\det\left[\int_a^b p(x)\,g_i(x)\,g_j(x)\,dx\right]\\
&\quad-\left(\det\left[\int_a^b p(x)\,f_i(x)\,g_j(x)\,dx\right]\right)^2\\
&\leq\frac{1}{4}(\Gamma-\gamma)^2\left(\det\left[\int_a^b p(x)\,g_i(x)\,g_j(x)\,dx\right]\right)^2.
\end{aligned}\tag{20}
$$

Proof. Applying the Pólya and Szegö equality B. Mond, J. Pečarić and B. Tepeš proved in [3] that the following result is valid if the hypothesis of Theorem 6 is fulfilled:

$$
(\gamma+\Gamma)\det\left[\int_a^b p(x)\,f_i(x)\,g_j(x)\,dx\right]
$$

$$\geq \det\left[\int_a^b p(x)\, f_i(x)\, f_j(x)\, dx\right] + \gamma\Gamma \det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right]. \tag{21}$$

Note that $\det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right] \geq 0$. So multiplying (21) by $\det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right]$ we have

$$\begin{aligned}
&\det\left[\int_a^b p(x)\, f_i(x)\, f_j(x)\, dx\right] \det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right] \\
&\quad + \gamma\Gamma \left(\det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right]\right)^2 \\
&\leq (\gamma+\Gamma) \det\left[\int_a^b p(x)\, f_i(x)\, g_j(x)\, dx\right] \det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\det\left[\int_a^b p(x)\, f_i(x)\, f_j(x)\, dx\right] \det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right] \\
&\quad - \left(\det\left[\int_a^b p(x)\, f_i(x)\, g_j(x)\, dx\right]\right)^2 \\
&\leq (\gamma+\Gamma) \det\left[\int_a^b p(x)\, f_i(x)\, g_j(x)\, dx\right] \det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right] \\
&\quad - \gamma\Gamma \left(\det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right]\right)^2 - \left(\det\left[\int_a^b p(x)\, f_i(x)\, g_j(x)\, dx\right]\right)^2 \\
&= \frac{1}{4}(\Gamma-\gamma)^2 \left(\det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right]\right)^2 \\
&\quad - \left(\frac{\gamma+\Gamma}{2} \det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right]\right)^2 - \left(\det\left[\int_a^b p(x)\, f_i(x)\, g_j(x)\, dx\right]\right)^2 \\
&\leq \frac{1}{4}(\Gamma-\gamma)^2 \left(\det\left[\int_a^b p(x)\, g_i(x)\, g_j(x)\, dx\right]\right)^2.
\end{aligned}$$

$\square$

Theorem 7 *Let $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ be vectors of real inner product space X with finite dimension $\dim X = n$ end $e_1, \ldots, e_n$ be any orthonormal base in*

X. Assume also that $\det\left[(x_i, e_{j_k})\right]$ *for* $r < n$ *and* $1 \leq j_1 < \ldots < j_r \leq n$ *satisfy the inequality*

$$m\det\left[(y_i, e_{j_k})\right] \leq \det\left[(x_i, e_{j_k})\right] \leq M\det\left[(y_i, e_{j_k})\right], \tag{22}$$

where m, M *are given positive real numbers. With Gram determinants* $\Gamma(x_1, \ldots, x_r) = \det\left[(x_i, x_j)\right]$ *and* $\Gamma(y_1, \ldots, y_r) = \det\left[(y_i, y_j)\right]$ *we have the inequality*

$$\begin{aligned}0 &\leq \Gamma(x_1, \ldots, x_r)\,\Gamma(y_1, \ldots, y_r) - \left(\det\left[(x_i, y_j)\right]\right)^2 \\ &\leq \frac{1}{4}(M-m)^2\left(\Gamma(y_1, \ldots, y_r)\right)^2.\end{aligned} \tag{23}$$

Proof. The first inequality in (23) is generalization of Cauchy's inequality [4, p. 599]. Let us note that B. Mond, J. Pečarić and B. Tepeš have proved in [3] that the following result is valid if the assumption of Theorem 7 is fulfilled:

$$(m+M)\det\left[(x_i, y_j)\right] \geq \Gamma(x_1, \ldots, x_r) + mM\Gamma(y_1, \ldots, y_r). \tag{24}$$

Note that $\Gamma(y_1, \ldots, y_r) \geq 0$. So multiplying (23) by $\Gamma(y_1, \ldots, y_r)$ we have

$$\begin{aligned}&\Gamma(x_1, \ldots, x_r)\,\Gamma(y_1, \ldots, y_r) + mM\left(\Gamma(y_1, \ldots, y_r)\right)^2 \\ &\leq (m+M)\det\left[(x_i, y_j)\right]\Gamma(y_1, \ldots, y_r).\end{aligned}$$

Therefore

$$\begin{aligned}&\Gamma(x_1, \ldots, x_r)\,\Gamma(y_1, \ldots, y_r) - \left(\det\left[(x_i, y_j)\right]\right)^2 \\ &\leq (m+M)\det\left[(x_i, y_j)\right]\Gamma(y_1, \ldots, y_r) - mM\left(\Gamma(y_1, \ldots, y_r)\right)^2 \\ &\quad - \left(\det\left[(x_i, y_j)\right]\right)^2 \\ &= \frac{1}{4}(M-m)^2\left(\Gamma(y_1, \ldots, y_r)\right)^2 - \left(\frac{m+M}{2}\det\left[\int_a^b p(x)\,g_i(x)\,g_j(x)\,dx\right]\right)^2 \\ &\quad - \left(\det\left[(x_i, y_j)\right]\right)^2 \\ &\leq \frac{1}{4}(M-m)^2\left(\Gamma(y_1, \ldots, y_r)\right)^2.\end{aligned}$$

∎

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Analysis of Operator Splitting for the Simulation of Advection-Diffusion in Unbounded Domains

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Abstract

A general methodology for the derivation of artificial boundary conditions (ABCs) for numerical solution to differential problems in unbounded domains is developed and applied to the linear multidimensional advection-diffusion-reaction equation. The approach is mainly based on the coupled use of the method of operator splitting and spline approximations for the initial data. Due to the temporal and spatial locality of the techniques involved the derived ABCs appear to be local, exact, and geometrically universal. Rigorous mathematical analysis of the resulting boundary value problems is performed.

Key words: Artificial boundary conditions, advection-diffusion-reaction equation, operator splitting, spline approximations, existence and stability of weak solutions, computational efficiency of algorithms.

AMS subject classification: 35D05, 35D10, 35K15, 35K20, 35Q35, 65M99.

1 INTRODUCTION

When numerically solving a differential problem originally formulated in an unbounded domain as a Cauchy problem, one has to reformulate it as a boundary value problem (BVP) for a finite computational domain. Therefore, it arises the question of imposing adequate boundary conditions. These conditions are often called absorbing, or transparent, or artificial boundary conditions (ABCs), and they are to satisfy the two requirements: First, they must lead to a well-posed BVP; second, the error between the solutions to the original Cauchy problem and the resulting BVP should be as small as possible in the computational domain; upon this, the most desired ABC is the *exact* boundary condition, i.e. when the error between the solutions is identically zero. It is important to emphasise that the boundary that surrounds the computational domain is *artificial*. In other words, it appears merely due to limitations on the computer memory rather than from the original formulation. Therefore, the question of imposing adequate boundary conditions is not trivial because one has no obvious way how to close the problem on the artificial boundary taking into consideration the afore-

said requirements. Moreover, apart from the adequacy the ABCs should match some additional properties which may be dictated by the computational restrictions or by the practical necessities coming from the problem to be studied. Specifically, from the computational point of view all the operators describing the solution at the artificial boundary must be local (i.e. differential, non-integral)—otherwise the ABC will be computationally unrealisable. As for the practical aspects, a large number of problems arising in practice have to be solved in domains of drastically complex geometries, and so the ABC should be as geometrically uncritical (or flexible) as possible. Hence, the fulfillment of these requirements provides additional difficulties when constructing ABCs.

In the last thirty years there have been suggested a whole series of different methods for the construction of ABCs for solving various problems of mathematical physics [1, 2, 5, 6, 7, 10, 11, 13, 14, 15, 16, 17, 18, 20, 22, 29, 34, 35, 36, 37, 38, 39, 40]. Nevertheless, all the methods developed so far possess certain disadvantages inherited from the analytical techniques which they are based on. For example, the classical methods based on the application of the Fourier transform yield geometrically exigent artificial boundary conditions that can be used with a rectangular boundary only. On the other hand, geometrically uncritical methods lead to global ABCs (in space—for steady state problems, and also in time—for non-stationary ones), which are unrealisable from the computational standpoint.

Among many ABC-constructing methods, let us emphasise those based on the method of difference potentials (or the difference potentials method,

DPM) originally invented by Ryaben'kii [33] and further studied in a series of papers by Ryaben'kii and Tsynkov [34, 36, 37]. The key point of the DPM-based approach is to replace the problem outside the computational domain by an operator equation on its boundary. Hence, the corresponding ABCs involve, in fact, the apparatus of Green functions. Unlike all other approaches, this is absolutely uncritical to the geometry of computational domain, universal from the point of view of the applicability to solving PDEs of various types (elliptic, hyperbolic, etc.), and besides it leads to ABCs that are easy-to-implement with existing interior solvers [37]. However, the DPM-based boundary conditions appear to be global, and so their numerical implementation requires a localisation of the operator at the boundary, which, obviously, implies some loss in the accuracy of solution.

Concerning all the existing methods for the construction of ABCs we should also notice that no one of the corresponding analytical techniques underlain takes into account the mathematical structure of the equations which it is applied to. This circumstance, in our opinion, essentially determines that the methods suffer from some or other disadvantages mentioned above, like, e.g., globality or low accuracy. Besides, many of the techniques are global in space, which also makes the boundary conditions possess the aforesaid restrictions. To clarify, let us consider, for instance, the linear homogeneous advection-diffusion equation with constant coefficients

$$\frac{\partial z}{\partial t} + \mathbf{U} \cdot \nabla z - D\nabla^2 z = 0 \tag{1}$$

subject to the boundary condition at infinity

$$\lim_{|x|+|y|\to+\infty} z(x,y,t) = 0 \quad \forall t > 0. \tag{2}$$

Let the computational domain be the half plane $x \leq 0$, and hence $x = 0$ is the boundary point. Applying to (1) the Fourier transform in y and in t, finding then the general solution in x, satisfying the boundary condition (2), and applying the inverse Fourier transform, we come to the following expression:

$$\left.\frac{\partial z}{\partial x}\right|_{x=0} = \int_{\mathbb{R}^2} P(\eta,\omega)\tilde{z}(0,\eta,\omega)e^{i(\eta y+\omega t)}d\eta d\omega. \tag{3}$$

Formula (3) represents the exact artificial boundary condition at the point $x = 0$ [17]. It can be seen that because of the presence of the factor $P(\eta,\omega)$ the integrals on the right-hand side of (3) cannot be inverted explicitly. Thus, the ABC appears to be global (due to the necessity of integration over the entire $\mathbb{R}^2$), and it is required to be somehow localised for numerical implementation. In this connection it is important to emphasise that ABC (3) was derived with the help of the Fourier transform applied to the *whole, original* equation. It can be seen, however, that equation (1) describes two different physical processes given by the advective and diffusive terms. Hence, an advanced method for the construction of ABCs could imply preliminary splitting of the original equation in time [19, 28], that is

$$\frac{\partial z_1}{\partial t} + \mathbf{U}\cdot\nabla z_1 = 0,$$
$$\frac{\partial z_2}{\partial t} - D\nabla^2 z_2 = 0.$$

The time splitting allows to take into account the mathematical structure of the equation to be studied. Specifically, due to the splitting we have reduced the complexity of the original equation (under the complexity we understand the number of independent physical processes considered simultaneously), and so the derivation of boundary conditions can now be done by those analytical techniques which could not be used before. As we shall see below, such a splitting coupled with some other methods may provide substantial benefits in the geometrical flexibility, accuracy, and locality of the resulting ABCs.

The aim of this paper is to investigate the method of splitting as an approach to the construction of highly precise, geometrically universal, and computationally implementable artificial boundary conditions for *multi*-processed, *multi*-dimensional equations of mathematical physics. It is important to note that although this method has already been intensively studied for the last forty years (see, e.g., [3, 19, 26, 28] and the bibliography there), so far it has not yet been employed within the framework of constructing ABCs [36]. More precisely, some aspects of operator splitting were analysed in the papers by Oliger and Sundström [30] and by Gustafsson and Sundström [14]. In particular, in [14], treating some problems of fluid dynamics, the authors represent the ABC as a mixed Dirichlet-Neumann boundary condition, where the first term (the Dirichlet's) corresponds to the purely transport process, while the second one (the Neumann's) relates to the viscosity only. Such a representation may, of course, be interpreted as splitting by physical processes. Nevertheless, unlike [14, 30] in the current paper we study the method of splitting

when it is applied to the original equation itself rather than to the boundary operator to be constructed. Besides, we also investigate dimensional splitting that has not been analysed so far in the context of ABCs. We shall develop a general ABC-methodology for the linear advection-diffusion-reaction equation that has many practical applications, e.g., in environmental modelling [27, 31]. The main focus of the paper will be on the analytical aspects of the method, while purely numerical issues will be out of our scope. The interested reader may refer, however, to work [9] where the numerical results are presented in detail.

The paper is organised as follows.

In section 2 we begin from the formulation of the original Cauchy problem and then state the main objective of the further research.

In section 3 we perform the time splitting of the original problem and demonstrate that it is valid in time in the small. Whereas those results are not new, we shall need them to refer to from other sections.

Section 4 is directly devoted to the construction of artificial boundary conditions for the previously derived subproblems. For the advection equation we construct a characteristic-based ABC. For the diffusive boundary conditions we essentially employ the technique of dimensional splitting combined with spline approximations for the initial data. Due to the spatial locality of these methods (unlike, e.g., the Fourier transform in space) the obtained ABCs are local both in time and in space and uncritical to the shape of artificial boundary. Moreover, we show that the ABCs appear to be exact for both subproblems, as well as justify the use of the time splitting

taking into account the derived boundary conditions.

In section 5 we prove two theorems on well-posedness of the resulting BVPs. For this we construct energetic Hilbert spaces with appropriate norms involving the theory of generalised functions. Besides, we demonstrate that the derived ABCs are physically correct in the sense of non-negativity of solution.

In section 6 we resume the paper.

2 PROBLEM FORMULATION

Consider the n-dimensional advection-diffusion-reaction Cauchy problem with respect to the function $z = z(\mathbf{x}, t)$

$$\frac{\partial z}{\partial t} + \nabla \cdot (\mathbf{U} z) - \nabla \cdot (D \nabla z) + \gamma z = f, \quad (\mathbf{x}, t) \in \mathbb{R}^n \times (0, +\infty), \tag{4}$$

$$z(\mathbf{x}, t)\Big|_{t=0} = g(\mathbf{x}), \tag{5}$$

$$\lim_{|\mathbf{x}| \to +\infty} z(\mathbf{x}, t) = 0 \quad \forall t > 0. \tag{6}$$

Here ∇ denotes the gradient operator, $\mathbf{U}(\mathbf{x}, t) = (\mathrm{u}_1(\mathbf{x}, t), \ldots, \mathrm{u}_n(\mathbf{x}, t))^{\mathrm{T}}$ is the vectorial field of velocities, $D = D(\mathbf{x}, t) > 0$ is the diffusion coefficient, $\gamma = \gamma(\mathbf{x}, t) \geq 0$ is the absorption coefficient, and $f = f(\mathbf{x}, t)$ on the right-hand side of (4) denotes the sources. The field $\mathbf{U}$ is supposed to be non-divergent, that is

$$\nabla \cdot \mathbf{U} = 0. \tag{7}$$

Let the domain of interest be an open region $\Omega \subset \mathbb{R}^n$ bounded by a piecewise smooth boundary Γ. Let also the parameters $\mathbf{U}$ and D be constant outside

Ω, as well as $f = 0$ in $\mathbb{R}^n \backslash \Omega$. Concerning Γ we additionally assume that it satisfies the condition

$$\forall \mathbf{x} \in \Gamma^- \quad \mathbf{x} - \mathbf{U}\vartheta \notin \Omega \quad \forall \vartheta \geq 0. \tag{8}$$

Here Γ^- is the inflow part of the boundary defined as $\Gamma^- := \{\mathbf{x} \in \Gamma : \mathbf{U} \cdot \mathbf{n} \leq 0\}$, while $\mathbf{n}$ is the outward unit normal to Γ. Property (8) will mainly be needed in section 4.1.

We want to reformulate problem (4)-(7) as a boundary value problem for the closed domain $\overline{\Omega} = \Omega \bigcup \Gamma$. Hence, it is required to replace (6) by a boundary condition of the form $Bz(\mathbf{x}, t)\Big|_{\mathbf{x} \in \Gamma} = 0$, where B is some operator. Upon this, we stipulate the following requirements:

1. The resulting boundary value problem

$$\frac{\partial z}{\partial t} + \nabla \cdot (\mathbf{U}z) - \nabla \cdot (D\nabla z) + \gamma z = f, \quad (\mathbf{x}, t) \in \Omega \times (0, +\infty), \tag{9}$$

$$z(\mathbf{x}, t)\Big|_{t=0} = g(\mathbf{x}), \tag{10}$$

$$Bz(\mathbf{x}, t)\Big|_{\mathbf{x} \in \Gamma} = 0, \quad t > 0, \tag{11}$$

$$\nabla \cdot \mathbf{U} = 0 \tag{12}$$

   must be well-posed in the sense of existence, uniqueness, and stability of solution.

2. The boundary operator B must be local (i.e. differential) both in time and in space.

Aside from these two stipulations, we are interested to construct B in such a manner that the difference between the solutions to problems (4)-(7) and (9)-(12) would be as small as possible in the domain Ω.

3 TIME SPLITTING

Before constructing the operator B let us return to the original problem (4)-(7).

It is important to observe that from the mathematical point of view equation (4) is composed of the advective, diffusive, and reactive terms. In other words, it describes three different physical processes given by the transport, diffusion, and reaction equations, respectively. Therefore, (4) can be split in time as follows [26, 28]:

$$\frac{\partial z_1}{\partial t} + \nabla \cdot (\mathbf{U} z_1) = 0, \tag{13}$$

$$\frac{\partial z_2}{\partial t} - \nabla \cdot (D \nabla z_2) = 0, \tag{14}$$

$$\frac{\partial z_3}{\partial t} + \gamma z_3 = f. \tag{15}$$

(Without loss of generality we write the sources on the right-hand side of (15) only.) For the particular case when $\mathbf{U}$, D and γ are constant and $f = 0$, it can be shown analytically that the sequential solution to equations (13)-(15) is equal to the solution to the original equation (4).

Indeed, denote by $Z(\boldsymbol{\xi}, t)$ and $G(\boldsymbol{\xi})$ the Fourier coefficients of the functions z and g, respectively, and consider the Fourier integrals

$$z(\mathbf{x}, t) = \int_{\mathbb{R}^n} Z(\boldsymbol{\xi}, t) e^{i \boldsymbol{\xi} \cdot \mathbf{x}} d\boldsymbol{\xi}, \tag{16}$$

$$g(\mathbf{x}) = \int_{\mathbb{R}^n} G(\boldsymbol{\xi}) e^{i \boldsymbol{\xi} \cdot \mathbf{x}} d\boldsymbol{\xi}. \tag{17}$$

Here $i^2 = -1$. Then the solution to the original, unsplit problem (4)-(7) has

the form

$$z(\mathbf{x},t) = \int_{\mathbb{R}^n} G(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot(\mathbf{x}-\mathbf{U}t)-t(D\boldsymbol{\xi}\cdot\boldsymbol{\xi}+\gamma)} d\boldsymbol{\xi}. \tag{18}$$

Consider now equations (13)-(15) and denote by $Z_j(\boldsymbol{\xi},t)$ the Fourier coefficients of the functions z_j ($j=\overline{1,3}$), respectively. Then it holds

$$z_1(\mathbf{x},t) = \int_{\mathbb{R}^n} Z_1(\boldsymbol{\xi},t) e^{i\boldsymbol{\xi}\cdot\mathbf{x}} d\boldsymbol{\xi}, \tag{19}$$

$$z_2(\mathbf{x},t) = \int_{\mathbb{R}^n} Z_2(\boldsymbol{\xi},t) e^{i\boldsymbol{\xi}\cdot\mathbf{x}} d\boldsymbol{\xi}, \tag{20}$$

$$z_3(\mathbf{x},t) = \int_{\mathbb{R}^n} Z_3(\boldsymbol{\xi},t) e^{i\boldsymbol{\xi}\cdot\mathbf{x}} d\boldsymbol{\xi}. \tag{21}$$

Now, following the concept of time splitting, for $t=\tau$ we first solve equation (13) with the initial condition $z_1(\mathbf{x},t)\Big|_{t=0} = g(\mathbf{x})$, then solve (14) with $z_2(\mathbf{x},t)\Big|_{t=0} = z_1(\mathbf{x},t)\Big|_{t=\tau}$, and finally solve (15) assuming $z_3(\mathbf{x},t)\Big|_{t=0} = z_2(\mathbf{x},t)\Big|_{t=\tau}$. Consequently, for the Fourier coefficients Z_j's we have ($\boldsymbol{\xi}$ is fixed)

$$Z_1(\boldsymbol{\xi},t) = G(\boldsymbol{\xi}) e^{-it\boldsymbol{\xi}\cdot\mathbf{U}}, \tag{22}$$

$$Z_2(\boldsymbol{\xi},t) = Z_1(\boldsymbol{\xi},\tau) e^{-Dt\boldsymbol{\xi}\cdot\boldsymbol{\xi}}, \tag{23}$$

$$Z_3(\boldsymbol{\xi},t) = Z_2(\boldsymbol{\xi},\tau) e^{-\gamma t}. \tag{24}$$

Substitution of (22) into (23) and further (23) into (24) yields

$$Z_3(\boldsymbol{\xi},t) = G(\boldsymbol{\xi}) e^{-i\tau\boldsymbol{\xi}\cdot\mathbf{U}-\tau(D\boldsymbol{\xi}\cdot\boldsymbol{\xi}+\gamma)}, \tag{25}$$

so that

$$z_3(\mathbf{x},\tau) = \int_{\mathbb{R}^n} G(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot(\mathbf{x}-\mathbf{U}\tau)-\tau(D\boldsymbol{\xi}\cdot\boldsymbol{\xi}+\gamma)} d\boldsymbol{\xi}. \tag{26}$$

It can be seen that expression (26) coincides with (18) at $t = \tau$.

Remark 1. Let us note that if the parameters $\mathbf{U}$, D and γ are non-constant or if $f \neq 0$ (which is, in fact, the case when solving advection-diffusion-reaction problems with the real data—see, e.g., [27, 31]), then the sequential solution to the split equations (13)-(15) does not exactly coincide with the original solution z, and in this case we have the approximate equality $z(\mathbf{x}, \tau) \approx z_3(\mathbf{x}, \tau)$. Obviously, the smaller the timestep τ the more accurate the solution z_3. This means that the splitting is valid in time *in the small* [27, 28].

Basing on the analytical calculations presented above, instead of the derivation of the boundary condition (11) for the whole advection-diffusion-reaction equation in the next section we shall first perform the time splitting of (4) and then construct two boundary operators, B_a and B_d, for the advective and diffusive subproblems, respectively. (It is clear that the reactive subproblem does not require any boundary condition because equation (15) has no spatial derivatives.) It is important to emphasise, however, that formulas (16)-(26) justify the time splitting when it is applied to the *Cauchy problem*, i.e. when $\mathbf{x}$ belongs to the whole space $\mathbb{R}^n$. Nevertheless, the use of the splitting within the framework of solution to *boundary value problems* requires to be substantiated additionally [3, 4]. Thus, in section 4 we shall also have to justify the time splitting taking into account the boundary operators B_a and B_d.

4 DERIVATION OF THE BOUNDARY OPERATORS

In this section we shall construct artificial boundary conditions for the advective and diffusive subproblems. For the sake of generality we shall omit the indices "1" and "2" at the function z, as well as assume the right-hand sides of the equations to be non-zero.

4.1 Advection

Let us consider the advective Cauchy problem

$$\frac{\partial z}{\partial t} + \nabla \cdot (\mathbf{U}z) = f, \quad (\mathbf{x}, t) \in \mathbb{R}^n \times (0, +\infty), \tag{27}$$

$$z(\mathbf{x}, t)\Big|_{t=0} = g(\mathbf{x}), \tag{28}$$

$$\lim_{|\mathbf{x}|\to+\infty} z(\mathbf{x}, t) = 0 \quad \forall t > 0, \tag{29}$$

$$\nabla \cdot \mathbf{U} = 0. \tag{30}$$

Under the assumption that the field $\mathbf{U}$ is constant and $f = 0$ in $\Sigma := \mathbb{R}^n \backslash \Omega$ (see section 2), we can rewrite (27)-(30) for the domains Ω and Σ as follows:

$$\frac{\partial z}{\partial t} + \nabla \cdot (\mathbf{U}z) = f, \quad (\mathbf{x}, t) \in \Omega \times (0, +\infty), \tag{31}$$

$$z(\mathbf{x}, t)\Big|_{t=0} = g(\mathbf{x}), \tag{32}$$

$$B_a z(\mathbf{x}, t)\Big|_{\mathbf{x}\in\Gamma} = 0, \quad t > 0, \tag{33}$$

$$\nabla \cdot \mathbf{U} = 0; \tag{34}$$

and

$$\frac{\partial z}{\partial t} + \mathbf{U} \cdot \nabla z = 0, \quad (\mathbf{x}, t) \in \Sigma \times (0, +\infty), \tag{35}$$

$$z(\mathbf{x}, t)\Big|_{t=0} = g(\mathbf{x}), \tag{36}$$

$$B_a z(\mathbf{x}, t)\Big|_{\mathbf{x} \in \Gamma} = 0, \quad t > 0, \tag{37}$$

$$\lim_{|\mathbf{x}| \to +\infty} z(\mathbf{x}, t) = 0 \quad \forall t > 0. \tag{38}$$

(In the second problem we omitted the divergence-free condition for $\mathbf{U}$ since it holds identically.) In order to close problem (31)-(34) in the domain $\overline{\Omega} = \Omega \bigcup \Gamma$, we have to specify the boundary operator B_a from (33) only on the inflow part of the boundary.

The derivation of B_a is straightforward: The solution to problem (35)-(38) on the set $\Gamma^- \subset \Sigma$ is determined by the formula

$$z_\Sigma(\mathbf{x}, t)\Big|_{\mathbf{x} \in \Gamma^-} = g(\mathbf{x} - \mathbf{U}t)\Big|_{\mathbf{x} \in \Gamma^-}, \tag{39}$$

and hence, taking into account the continuity condition $z_\Omega(\mathbf{x}, t)\Big|_{\mathbf{x} \in \Gamma^-} = z_\Sigma(\mathbf{x}, t)\Big|_{\mathbf{x} \in \Gamma^-}$, we can write

$$B_a = \bullet - g(\mathbf{x} - \mathbf{U}t)\Big|_{\mathbf{x} \in \Gamma^-}. \tag{40}$$

Here the subscripts Ω and Σ at the function z are to distinguish the solutions to problems (31)-(34) and (35)-(38), respectively, while the point "$\bullet$" denotes the place of $z_\Omega(\mathbf{x}, t)\Big|_{\mathbf{x} \in \Gamma^-}$.

Expression (40) is the exact local artificial boundary condition for problem (31)-(34).

Remark 2 (on characteristic-based ABCs and time splitting). It is well-known that ABC (40) is a standard characteristic-based boundary condition typically used when solving the linear *first*-order *advection* equation (31) (some discussions on characteristic-based ABCs can be found in [36], pp. 491-498). However, it is important to emphasise that due to the splitting we derived (40) for solving the *second*-order, *multi*-processed advection-*diffusion*-*reaction* equation (9). Obviously that without the preliminary splitting of (4) in time the characteristic-based boundary condition could not be applied before. A substantial advantage of this approach is that having split (4) we were able to obtain the *exact* and simultaneously *local* ABC (unlike, say, [16, 17]). Furthermore, upon this we did not impose essential restrictions on the shape of the boundary Γ (the only restriction is condition (8)), and so the boundary condition appeared to be geometrically flexible.

Remark 3. From (40) it becomes clear the necessity of condition (8) from section 2. Indeed, if (8) does not hold then solution (39) is only valid until some $t^* < +\infty$ (in fact, for a fixed $\mathbf{x} \in \Gamma^-$ we have $t^* = \min\limits_{\mathbf{x}-\mathbf{U}\vartheta\in\Omega} \vartheta$), so that the applicability of (39) (and therefore, of (40)) is restricted in time, unless we specify the boundary operator B_a of problem (35)-(38) on the outflow part $\Gamma^+ := \Gamma\backslash\Gamma^-$ of the boundary. However, under (8) formulas (39) and (40) are valid for any $t > 0$.

4.2 Diffusion

Consider the diffusive Cauchy problem

$$\frac{\partial z}{\partial t} - \nabla \cdot (D\nabla z) = f, \quad (\mathbf{x}, t) \in \mathbb{R}^n \times (0, +\infty), \tag{41}$$

$$z(\mathbf{x}, t)\Big|_{t=0} = g(\mathbf{x}), \tag{42}$$

$$\lim_{|\mathbf{x}|\to+\infty} z(\mathbf{x}, t) = 0 \quad \forall t > 0. \tag{43}$$

Analogously to the advective case, we have to reformulate it as

$$\frac{\partial z}{\partial t} - \nabla \cdot (D\nabla z) = f, \quad (\mathbf{x}, t) \in \Omega \times (0, +\infty), \quad \mathrm{supp} f \subseteq \Omega, \tag{44}$$

$$z(\mathbf{x}, t)\Big|_{t=0} = g(\mathbf{x}), \tag{45}$$

$$B_d z(\mathbf{x}, t)\Big|_{\mathbf{x}\in\Gamma} = 0, \quad t > 0. \tag{46}$$

We shall construct the boundary operator B_d in two steps. At the first step we shall assume the coefficient D to be frozen and $f = 0$ in the whole space $\mathbb{R}^n$, and derive ABCs for the use with the homogeneous equation (44) with constant diffusion coefficient. At the second step we shall justify the possibility of using the constructed ABCs with the inhomogeneous equation (44) and variable parameter D.

Let us consider the Cauchy problem

$$\frac{\partial z}{\partial t} - D\nabla \cdot \nabla z = f, \quad (\mathbf{x}, t) \in \mathbb{R}^n \times (0, +\infty), \tag{47}$$

$$z(\mathbf{x}, t)\Big|_{t=0} = g(\mathbf{x}), \tag{48}$$

$$\lim_{|\mathbf{x}|\to+\infty} z(\mathbf{x}, t) = 0 \quad \forall t > 0. \tag{49}$$

Applying to (47)-(49) the Laplace transform in time, in the dual space we

obtain

$$p\mathcal{L}[z] - D\nabla \cdot \nabla \mathcal{L}[z] = g, \quad (\mathbf{x}, p) \in \mathbb{R}^n \times \mathbb{C}, \tag{50}$$

$$\lim_{|\mathbf{x}| \to +\infty} \mathcal{L}[z] = 0. \tag{51}$$

Here $\mathcal{L}[z]$ denotes the image of the function z, whereas p is the parameter of the Laplace transform. Now, in order to find the solution to problem (50)-(51), we involve the technique of dimensional splitting [19, 28]. Namely, we choose some k from the range $[1, \ldots, n]$ and consider problem (50)-(51) with respect to x_k only, leaving all the other x_l's ($l \neq k$) fixed. This will allow to reduce the spatial dimensionality of equation (50) from n to 1 and to find an analytical expression for the solution without employing other techniques that may introduce undesired restrictions on the applicability of the subsequent ABCs. So, we have

$$p(\mathcal{L}[z])_{x_k} - D\frac{d^2(\mathcal{L}[z])_{x_k}}{dx_k^2} = g_{x_k}, \quad (x_k, p) \in \mathbb{R} \times \mathbb{C}, \tag{52}$$

$$\lim_{|x_k| \to +\infty} (\mathcal{L}[z])_{x_k} = 0, \tag{53}$$

where $(\mathcal{L}[z])_{x_k} \equiv (\mathcal{L}[z])\Big|_{x_k \in \mathbb{R}, \{x_l\}_{l=1, l\neq k}^n \in \mathbb{R}^{n-1}}$ and $g_{x_k} \equiv g\Big|_{x_k \in \mathbb{R}, \{x_l\}_{l=1, l\neq k}^n \in \mathbb{R}^{n-1}}$. Expression (52) is a linear inhomogeneous second-order ordinary differential equation with constant coefficients; its solution, given the boundary condition (53), has the form

$$(\mathcal{L}[z])_{x_k} = \frac{1}{D(w_2 - w_1)} \left(e^{w_1 x_k} \int g_{x_k} e^{-w_1 x_k} dx_k - e^{w_2 x_k} \int g_{x_k} e^{-w_2 x_k} dx_k \right), \tag{54}$$

where $w_{1,2} = \pm\sqrt{\frac{p}{D}}$.

It should be stressed that an explicit representation of (54) requires calculating the indefinite integrals of the function g multiplied with the exponents

$e^{-w_1 x_k}$ and $e^{-w_2 x_k}$. It is clear, however, that the initial condition g (and hence, its reduction g_{x_k}) can be an arbitrary function, and so the integrals in (54) may be rather cumbersome for analytical calculations or even inexpressible in elementary functions. Therefore, we introduce an infinite partition $\Pi = \{\pi_{-\infty}, \ldots, \pi_{N-1}, \pi_N, \pi_{N+1}, \ldots, \pi_{+\infty}\}$ of the interval $(-\infty, +\infty)$, and in every segment $[\pi_{N-1}, \pi_N] \subset \Pi$ ($N \in \mathbb{Z}$) we approximate g_{x_k} by a spline of order three:

$$g_{x_k}\Big|_{x_k \in [\pi_{N-1}, \pi_N]} \approx \sum_{j=0}^{3} a_j^{(N)} x_k^j, \quad a_j^{(N)} \in \mathbb{R}. \tag{55}$$

Then substitution of (55) into (54) after a series of manipulations will yield

$$(\mathcal{L}[z])_{x_k} \approx (\mathcal{L}[z])_{x_k}^{(3)} = \frac{1}{p} g_{x_k} + \frac{D}{p^2} \frac{d^2 g_{x_k}}{dx_k^2},$$

and by the tables of the Laplace transform we derive the solution in the original space

$$z_{x_k} \approx z_{x_k}^{(3)} = g_{x_k} + Dt \frac{d^2 g_{x_k}}{dx_k^2}.$$

Here the superscript (3) denotes the spline order, while $z_{x_k} \equiv z\Big|_{x_k \in \mathbb{R}, \{x_l\}_{l=1, l \neq k}^n \in \mathbb{R}^{n-1}}$. Finally, employing again the technique of dimensional splitting, we find the solution in all n directions:

$$z(\mathbf{x}, t) \approx z^{(3)}(\mathbf{x}, t) = g(\mathbf{x}) + Dt \sum_{l=1}^{n} \frac{\partial^2 g(\mathbf{x})}{\partial x_l^2}.$$

Note that if the function g is sufficiently smooth then for an arbitrary odd spline order $2m+1$ ($m \in \mathbb{N}$) it holds

$$z(\mathbf{x}, t) \approx z^{(2m+1)}(\mathbf{x}, t) = g(\mathbf{x}) + \sum_{r=1}^{m} D^r \frac{t^r}{r!} \sum_{l=1}^{n} \frac{\partial^{2r} g(\mathbf{x})}{\partial x_l^{2r}}. \tag{56}$$

It should also be remarked that the same solutions are obtained when using splines of even orders, so that the property "even/odd" does not produce an effect on the final result.

Expression (56) is an infinite family of approximate solutions to the homogeneous Cauchy problem (47)-(49) with constant parameter D.

Let us make several important comments.

First of all, concerning (56) one may easily observe that for every $m \in \mathbb{N}$, at a fixed $\mathbf{x} \in \mathbb{R}^n$, the corresponding solution $z^{(2m+1)}(\mathbf{x}, t)$ tends to infinity when $t \to +\infty$, which contradicts the temporal asymptotics of the diffusive process. Nevertheless, for a finite $t = \tau$ the function $z^{(2m+1)}(\mathbf{x}, t)\Big|_{t=\tau}$ can be recognised as a finite difference solution to the temporally discretised diffusion equation (47). Indeed, in case of $m = 1$ formula (56) is simply the Euler method (or the method of Runge-Kutta of order one); in case of $m = 2$ we have the midpoint method (or the Runge-Kutta of order two); the case of $m = 4$ corresponds to the classical method of Runge-Kutta of order four, and so on [32]. Consequently, for each $m \in \mathbb{N}$ the corresponding solution (56) is admissible with any *finite* timestep $\tau_j = t_{j+1} - t_j$, so that we must rewrite (56) in the form

$$z^{[j+1]}(\mathbf{x}, t) \approx z^{[j]}(\mathbf{x}, t) + \sum_{r=1}^{m} D^r \frac{\tau_j^r}{r!} \sum_{l=1}^{n} \frac{\partial^{2r} z^{[j]}(\mathbf{x}, t)}{\partial x_l^{2r}},$$
$$z^{[0]}(\mathbf{x}, t) \equiv g(\mathbf{x}), \quad j \in \{0\} \cup \mathbb{N}. \tag{57}$$

Thus, we have an infinite hierarchy of diffusive boundary operators defined

as

$$B_d^{(2m+1)} = \bullet - \left(z^{[j]}(\mathbf{x}, t) + \sum_{r=1}^{m} D^r \frac{\tau_j^r}{r!} \sum_{l=1}^{n} \frac{\partial^{2r} z^{[j]}(\mathbf{x}, t)}{\partial x_l^{2r}} \right) \Bigg|_{\mathbf{x} \in \Gamma},$$
$$z^{[0]}(\mathbf{x}, t) \equiv g(\mathbf{x}), \quad j \in \{0\} \cup \mathbb{N}. \tag{58}$$

Another remark concerns the possibility of using ABCs (58) with the inhomogeneous equation (44) and $D = D(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$. Specifically, due to the reduction of (56) to (57) it can be seen that the boundary conditions (58) coupled with a finite difference scheme employed to compute the solution in the interior do take into account the presence of sources on the right-hand side of equation (44) and the variability of the diffusion coefficient D in Ω too. Indeed, let, for example, $g(\mathbf{x}) = 0$. Then, in case of no reduction of (56) to (57), the direct employment of (56) would merely yield the zero Dirichlet boundary condition for any $t > 0$, and hence, it would totally disregard the presence of $f \neq 0$ in Ω. Nevertheless, in case of using (57) the solution $z^{[j+1]}(\mathbf{x}, t)$ on Γ may be equal to zero only until some t^*, when the perturbations of the solution in Ω caused by the presence of the non-zero right-hand side of (44) will achieve the boundary under the influence of diffusion. Therefore, for $t > t^*$ ABCs (57) (or (58), as well) stop being Dirichlet-like, just because $z^{[j]}(\mathbf{x}, t) \neq 0$ at the boundary points any longer. Hence, the sources f are really taken into account while reducing (56) to (57) [8, 9]. (Obviously, analogous ratiocinations can be repeated with an arbitrary initial condition $g(\mathbf{x})$.) As for the variable diffusion coefficient D in Ω, it is simply implemented by an appropriate interior finite difference scheme.

Let us note that Hagstrom and H. B. Keller in [18], Hagstrom in [15, 16],

as well as Sofronov in [35] use a similar approach for the derivation of ABCs. Namely, these authors first apply the separation of variables and then expand the solutions into Fourier series in basis functions of certain types. However, in doing so they consider one-dimensional problems (i.e. with respect to only one coordinate, leaving the others untreated), and do not derive ABCs for the original n-D equations, although the construction of *multi*-dimensional boundary conditions is a critical point because this can be done in different ways each of which may restrict the applicability of the subsequent ABCs. Moreover, in our opinion, a disadvantage of the ABCs constructed in [15, 16, 18, 35] is that the basis functions in the Fourier series were chosen to be *infinitely* supported (see the quoted papers for details), which imposes substantial restrictions on the geometry of the domain of interest, as well as implies the ABCs to be global. In this connection we stress that the function g in (55) was approximated by *compactly* supported interpolants, and therefore the subsequent boundary conditions appeared to be local. Furthermore, the use of dimensional splitting allowed to avoid employing geometrically exigent techniques (like, say, the Fourier transform in space—see formulas (1)-(3)) for the generalisation of the ABCs for an arbitrary $n > 1$, and so the n-D boundary conditions were obtained uncritical to the shape of the boundary Γ. This methodology may, of course, imply further studies on the use of various systems of compactly supported basis functions, e.g., some families of wavelets.

For the estimates of the errors produced by the boundary operators

$B_d^{(2m+1)}$ on Γ we have

$$\varepsilon_m^{[j+1]}(\mathbf{x},t) \equiv \left[\frac{\partial z(\mathbf{x},t)}{\partial t} - D\nabla\cdot\nabla z(\mathbf{x},t)\right]\Big|_{\mathbf{x}\in\Gamma} = \cdots =$$

$$= -D\left[D^m\frac{\tau_j^m}{m!}\sum_{l=1}^{n}\frac{\partial^{2m+2}z^{[j]}(\mathbf{x},t)}{\partial x_l^{2m+2}} + \sum_{r=1}^{m}D^r\frac{\tau_j^r}{r!}\sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n}\frac{\partial^{2r+2}z^{[j]}(\mathbf{x},t)}{\partial x_k^2\partial x_l^{2r}}\right]\Bigg|_{\mathbf{x}\in\Gamma}.$$

Denote $c_1(m) = \max_{\mathbf{x}\in\Gamma,l=\overline{1,n}}\left|\frac{\partial^{2m+2}z^{[j]}(\mathbf{x},t)}{\partial x_l^{2m+2}}\right|$, $c_2(m) = \max_{\mathbf{x}\in\Gamma,r=\overline{1,m},k,l=\overline{1,n},l\neq k}\left|\frac{\partial^{2r+2}z^{[j]}(\mathbf{x},t)}{\partial x_k^2\partial x_l^{2r}}\right|$. Then

$$\left|\varepsilon_m^{[j+1]}\right| \leq Dn\left(c_1\frac{(D\tau_j)^m}{m!} + c_2(n-1)\sum_{r=1}^{m}\frac{(D\tau_j)^r}{r!}\right),$$

and further, if c_1 and c_2 are bounded in m, it holds

$$\left|\varepsilon_\infty^{[j+1]}\right| \leq Dn(n-1)\left(e^{D\tau_j}-1\right)\sigma, \quad \sigma = \sup_{m\in\mathbb{N}} c_2(m).$$

4.3 Analysis of the time splitting for the resulting BVPs

In section 3, mainly basing on the results by Marchuk [28], we demonstrated that the sequential solution to the split advection-diffusion-reaction equation is equal to the solution to the original, unsplit problem. Nevertheless, in doing so we considered the case $\mathbf{x}\in\mathbb{R}^n$, i.e. we disregarded the presence of the boundary conditions. It is known, however, that while performing a time splitting there may occur spurious errors on the boundary if the subsequent boundary operators are defined in an inadequate manner (see, e.g., the classical example by D'yakonov [3, 4]). Therefore, now we shall show that the constructed ABCs do not introduce spurious errors into the solution on Γ.

The key point is that the constructed boundary conditions are *exact*, that is the errors between the solutions to the infinite-domain advective and diffusive problems and the corresponding BVPs are equal to zero. Indeed, concerning the advective problem, the operator B_a defined via (40) simply determines the first-order characteristic-based boundary condition, and so there is no error produced by B_a on Γ at all. As for the diffusive problem, above we observed that for every $m \in \mathbb{N}$ the corresponding function $z^{(2m+1)}(\mathbf{x}, t)$ is the exact solution to the temporally discretised diffusion equation

$$\frac{z^{[j+1]} - z^{[j]}}{\tau_j} - D\nabla \cdot \nabla z^{[j]} = 0, \quad z^{[0]} \equiv g, \quad \mathbf{x} \in \Gamma,$$
$$j \in \{0\} \cup \mathbb{N}. \tag{59}$$

For example, applying to (59) the fourth-order Runge-Kutta method in each direction, we have [32]

$$h_{1,x_l} = D\tau_j \frac{\partial^2 z^{[j]}}{\partial x_l^2}, \quad l = \overline{1,n},$$
$$h_{2,x_l} = D\tau_j \frac{\partial^2}{\partial x_l^2}\left(z^{[j]} + \frac{1}{2}h_{1,x_l}\right), \quad l = \overline{1,n},$$
$$h_{3,x_l} = D\tau_j \frac{\partial^2}{\partial x_l^2}\left(z^{[j]} + \frac{1}{2}h_{2,x_l}\right), \quad l = \overline{1,n},$$
$$h_{4,x_l} = D\tau_j \frac{\partial^2}{\partial x_l^2}\left(z^{[j]} + h_{3,x_l}\right), \quad l = \overline{1,n},$$

and so

$$z^{[j+1]} \approx z^{(RK4)} = z^{[j]} + \sum_{l=1}^{n}\left(\frac{h_{1,x_l}}{6} + \frac{h_{2,x_l}}{3} + \frac{h_{3,x_l}}{3} + \frac{h_{4,x_l}}{6}\right) = \cdots =$$
$$= z^{[j]} + D\tau_j \nabla^2 z^{[j]} + D^2\frac{\tau_j^2}{2}\sum_{l=1}^{n}\frac{\partial^4 z^{[j]}}{\partial x_l^4} +$$

$$+D^3\frac{\tau_j^3}{6}\sum_{l=1}^{n}\frac{\partial^6 z^{[j]}}{\partial x_l^6}+D^4\frac{\tau_j^4}{24}\sum_{l=1}^{n}\frac{\partial^8 z^{[j]}}{\partial x_l^8},$$

which is identically equal to (57) when $m=4$. Thus, for every $m\in\mathbb{N}$ the corresponding boundary operator $B_d^{(2m+1)}$ produces no error on Γ either. It should be stressed, however, that this statement holds for *discrete* times only, i.e. when the partial derivative $\frac{\partial z}{\partial t}$ is replaced by the approximation $\frac{z^{[j+1]}-z^{[j]}}{\tau_j}$. Obviously also that numerical implementation of (58) requires approximations of the spatial derivatives of the function $z^{[j]}(\mathbf{x},t)$, which may introduce discretisation errors [9]. However, in the continuous (in $\mathbf{x}$) case ABCs (58) remain exact.

Let us remark again that due to the splitting of equation (9) in time there does occur an error caused by the circumstance that the solution is now computed sequentially, with a (small) timestep τ. Nevertheless, as the operators B_a and $B_d^{(2m+1)}$ do not produce errors on Γ, the difference between the solutions to the original, unsplit equation and the split problems certainly tends to zero when $\tau\to 0$.

Consequently, since (i) the time splitting is justified for Cauchy problems (27)-(30) and (41)-(43), and (ii) the Cauchy problems are *exactly* reformulated as boundary value problems (31)-(34) and (44)-(46) with the boundary operators defined via (40) and (58), respectively, the use of the splitting is justified within the framework of solution to the resulting BVPs. In other words, the boundary operators B_a and $B_d^{(2m+1)}$ ($m\in\mathbb{N}$) are derived in a correct way.

5 ANALYSIS OF THE BOUNDARY VALUE PROBLEMS

5.1 Well-posedness of the advective problem

Let $0 \le t \le T < +\infty$. Making the change of variable $z(\mathbf{x}, t) = s(\mathbf{x}, t) + g(\mathbf{x} - \mathbf{U}t)$ we reduce BVP (31)-(34) with the operator B_a defined via (40) to

$$\frac{\partial s}{\partial t} + \nabla \cdot (\mathbf{U}s) = f_s, \quad (\mathbf{x}, t) \in \Omega \times (0, T), \tag{60}$$

$$s(\mathbf{x}, t)\Big|_{t=0} = 0, \tag{61}$$

$$s(\mathbf{x}, t)\Big|_{\mathbf{x} \in \Gamma^-} = 0, \quad 0 < t \le T, \tag{62}$$

$$\nabla \cdot \mathbf{U} = 0, \tag{63}$$

where

$$f_s = \begin{cases} f - \frac{\partial g(\mathbf{x} - \mathbf{U}t)}{\partial t} - \nabla \cdot (\mathbf{U} g(\mathbf{x} - \mathbf{U}t)), & \mathbf{x} \in \Omega, \\ -\frac{\partial g(\mathbf{x} - \mathbf{U}t)}{\partial t} - \nabla \cdot (\mathbf{U} g(\mathbf{x} - \mathbf{U}t)), & \mathbf{x} \notin \Omega. \end{cases}$$

It can be seen that problems (31)-(34) and (60)-(63) are identical.

Theorem 1. Problem (60)-(63) is well-posed in the sense of existence, uniqueness, and stability of solution.

To prove the theorem we shall need to specify some notation.

Let $Q = \Omega \times (0, T)$. Introduce the set

$$W := \left\{ s \in L_2(Q) : s \in C^1(\overline{Q}), s|_{t=0} = 0 \right\},$$

and define on W the norm

$$||s||_H := \frac{1}{\sqrt{2}}\sqrt{||s||_T^2 + \chi||s||_{\Gamma^+}^2} \tag{64}$$

and the scalar product

$$(s_1, s_2)_H := \frac{1}{2}\left[(s_1, s_2)_T + \chi(s_1, s_2)_{\Gamma^+}\right]. \tag{65}$$

Here $\overline{Q} = \overline{\Omega} \times [0, T]$, $\chi > 0$, while

$$\begin{aligned}
||s||_T &:= \sqrt{\int_\Omega |s(\mathbf{x}, T)|^2 \, d\Omega}, \\
||s||_{\Gamma^+} &:= \sqrt{\int_0^T\!\!\int_{\Gamma^+} |s(\mathbf{x}, t)|^2 \, d\Gamma^+ dt}, \\
(s_1, s_2)_T &:= \int_\Omega s_1(\mathbf{x}, T) s_2(\mathbf{x}, T) d\Omega, \\
(s_1, s_2)_{\Gamma^+} &:= \int_0^T\!\!\int_{\Gamma^+} s_1(\mathbf{x}, t) s_2(\mathbf{x}, t) d\Gamma^+ dt.
\end{aligned}$$

Let also $H(Q)$ denote the Hilbert space given by the pair $(\overline{W}, ||\cdot||_H)$.

Definition 1. A function $s \in H(Q)$ is said to be a generalised (or weak) solution to problem (60)-(63) if it satisfies the identity

$$\left(\frac{\partial s}{\partial t}, \psi\right) + (\nabla \cdot (\mathbf{U}s), \psi) = (f_s, \psi) \tag{66}$$

for any $\psi \in H(Q)$ [24].

Hence, we have to demonstrate that problem (60)-(63) has a unique generalised solution that continuously depends on the initial data.

Proof of Theorem 1. Existence. Let $a_j[s,\psi]$ $(j=1,2)$ denote the jth summand on the left-hand side of expression (66). Then it holds

$$|a_1[s,\psi]| \equiv \left|\left(\frac{\partial s}{\partial t},\psi\right)\right| \le \left\|\frac{\partial s}{\partial t}\right\| \|\psi\| < +\infty,$$
$$|a_2[s,\psi]| \equiv |(\nabla\cdot(\mathbf{U}s),\psi)| \le \beta_1\beta_2 n\,\|\psi\| < +\infty,$$

where $\beta_1 = \max_{(\mathbf{x},t)\in\overline{Q}} |\mathbf{U}| < +\infty$, $\beta_2 = \max_{l=\overline{1,n}} \left\|\frac{\partial s}{\partial x_l}\right\| < +\infty$, and so the a_j's are linear bounded functionals with respect to s. Consequently, in accordance with the Riesz theorem [21] we can write

$$a_j[s,\psi] = (s,v_j)_H \quad \forall s\in H(Q), \quad v_j\in H(Q), \quad j=1,2.$$

The correspondences $\psi\to v_j$ $(j=1,2)$ permit to introduce linear operators $L_j : H(Q)\to H(Q)$, so that

$$a_j[s,\psi] = (s,L_j\psi)_H \quad \forall s\in H(Q), \quad j=1,2.$$

In a similar way, for the right-hand side of (66) we have

$$a_3[f_s,\psi] = (v_3,\psi)_H \quad \forall\psi\in H(Q), \quad v_3\in H(Q).$$

Therefore, instead of (66) we obtain

$$(s,L\psi)_H = (v_3,\psi)_H,$$

where $L = L_1 + L_2$.

Lemma 1. The inverse operator $L^{-1} : R(H(Q))\to H(Q)$ exists and is bounded.

Proof. To prove the lemma we have to show that $\forall q \in H(Q)\ \ (q, Lq)_H \geq (q,q)_H$ [21].

Let $q \in H(Q)$. Then it holds

$$a_1[q,q] \equiv \left(\frac{\partial q}{\partial t}, q\right) = \frac{1}{2}\int_\Omega q^2(\mathbf{x}, T)d\Omega = \frac{1}{2}\left\|q\right\|_T^2, \tag{67}$$

$$\begin{aligned} a_2[q,q] \equiv (\nabla\cdot(\mathbf{U}q), q) = \frac{1}{2}\int_0^T\!\!\int_\Omega \nabla\cdot(\mathbf{U}q^2)d\Omega dt = \\ = \frac{1}{2}\int_0^T\!\!\int_{\Gamma^+}(\mathbf{U}\cdot\mathbf{n})q^2 d\Gamma^+ dt \geq \frac{\mu}{2}\left\|q\right\|_{\Gamma^+}^2. \end{aligned} \tag{68}$$

Here $0 < \mu = \min\limits_{(\mathbf{x},t)\in\Gamma^+\times[0,T]}(\mathbf{U}\cdot\mathbf{n})$. To obtain formula (67) we used condition (61), whereas for (68) we involved the divergence theorem [23] and conditions (62), (63). Choosing now χ in (65) such that $\chi < \mu$, we have

$$(q, Lq)_H = a_1[q,q] + a_2[q,q] \geq \frac{1}{2}\left\|q\right\|_T^2 + \frac{\mu}{2}\left\|q\right\|_{\Gamma^+}^2 \geq (q,q)_H.$$

The lemma is proved. ■

Let L^{-1} be the inverse operator to L and let $\tilde{L}^{-1}$ be such an extension of L^{-1} onto the whole space $H(Q)$ that $\tilde{L}^{-1}L = I$. By Lemma 1, L^{-1} exists and is bounded, and hence, its extension $\tilde{L}^{-1}$ exists and is bounded too. The operator $(\tilde{L}^{-1})^*$, adjoint to $\tilde{L}^{-1}$, is unique and has the same norm as $\tilde{L}^{-1}$. Therefore, $\forall q = L\psi \in H(Q)$ the identity

$$(s,q)_H = (s, L\psi)_H = (v_3, \psi)_H = (v_3, \tilde{L}^{-1}q)_H = ((\tilde{L}^{-1})^* v_3, q)_H$$

holds iff $s = (\tilde{L}^{-1})^* v_3$. Existence is proved.

Uniqueness. Let there exist two solutions, s_1 and s_2. Then for the

difference $\delta s = s_1 - s_2$ we have

$$\begin{aligned} \frac{\partial \delta s}{\partial t} + \nabla \cdot (\mathbf{U}\delta s) = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T), \\ \delta s(\mathbf{x}, t)\Big|_{t=0} = 0, \\ \delta s(\mathbf{x}, t)\Big|_{\mathbf{x}\in\Gamma^-} = 0, \quad 0 < t \leq T, \\ \nabla \cdot \mathbf{U} = 0. \end{aligned}$$

Substitution of δs into (66) with $\psi = \delta s$ yields

$$\frac{1}{2}\int_\Omega \delta s^2(\mathbf{x}, T) d\Omega + \frac{1}{2}\int_0^T\!\!\int_{\Gamma^+} (\mathbf{U}\cdot\mathbf{n})\delta s^2 d\Gamma^+ dt = 0. \tag{69}$$

Since the both summands on the left-hand side of (69) are positive, the identity holds iff $\delta s = 0$. Uniqueness is proved.

Stability. Let s_1 and s_2 be two different solutions that correspond to the two different sets of initial data $\{g_1, f_{s,1}\}$ and $\{g_2, f_{s,2}\}$, respectively. Denote $\delta s = s_1 - s_2$, $\delta f_s = f_{s,1} - f_{s,2}$, $\delta g = g_1 - g_2$, and substitute δs into (66) assuming that $\psi = \delta s$. Then

$$\frac{1}{2}\left\|\delta s\right\|_T^2 + \frac{\mu}{2}\left\|\delta s\right\|_{\Gamma^+}^2 \leq \frac{1}{2}\left\|\delta g\right\|_0^2 + \left\|\delta f_s\right\|\left\|\delta s\right\|, \tag{70}$$

where $\left\|\delta g\right\|_0 := \sqrt{\int_\Omega |\delta g(\mathbf{x})|^2 d\Omega}$. Since the left-hand side of (70) is bounded, there exist two constants, ς_1 and ς_2, such that $\varsigma_1 \geq \frac{\|\delta s\|_T}{2(\|\delta g\|_0 + \|\delta f_s\|)}$, $\varsigma_2 \geq \frac{\mu\|\delta s\|_{\Gamma^+}}{2(\|\delta g\|_0 + \|\delta f_s\|)}$, and so

$$\frac{1}{2}\left\|\delta s\right\|_T^2 + \frac{\mu}{2}\left\|\delta s\right\|_{\Gamma^+}^2 \leq \left\|\delta s\right\|_Q \left(\left\|\delta g\right\|_0 + \left\|\delta f_s\right\|\right), \tag{71}$$

where

$$\|\delta s\|_Q := \varsigma_1\|\delta s\|_T + \varsigma_2\|\delta s\|_{\Gamma^+}. \tag{72}$$

Now we observe that each summand on the left-hand side of (71) is not greater than the right-hand side—

$$\|\delta s\|_T \leq \sqrt{2\|\delta s\|_Q \left(\|\delta g\|_0 + \|\delta f_s\|\right)},$$
$$\|\delta s\|_{\Gamma^+} \leq \sqrt{\frac{2\|\delta s\|_Q \left(\|\delta g\|_0 + \|\delta f_s\|\right)}{\mu}},$$

and hence, given (72) we come to the estimate

$$\|\delta s\|_Q \leq K(\|\delta g\|_0 + \|\delta f_s\|),$$

where $K = 2(\varsigma_1 + \frac{\varsigma_2}{\sqrt{\mu}})^2$. The proof of the theorem is complete. ■

5.2 Well-posedness of the diffusive problem

Let $t_j \leq t \leq t_{j+1} < +\infty$. We make the change of variable $z^{[j+1]}(\mathbf{x}, t) = s(\mathbf{x}, t) + z^{[j]}(\mathbf{x}, t) + \sum_{r=1}^{m} D^r \frac{\tau_j^r}{r!} \sum_{l=1}^{n} \frac{\partial^{2r} z^{[j]}(\mathbf{x},t)}{\partial x_l^{2r}}$ and consider the problem

$$\frac{\partial s}{\partial t} - \nabla \cdot (D \nabla s) = f_s, \quad (\mathbf{x}, t) \in \Omega \times (t_j, t_{j+1}), \tag{73}$$

$$s(\mathbf{x}, t)\Big|_{t=t_j} = 0, \tag{74}$$

$$s(\mathbf{x}, t)\Big|_{\mathbf{x} \in \Gamma} = 0, \quad t_j < t \leq t_{j+1} \tag{75}$$

with the corresponding right-hand side f_s. (Here, unlike section 5.1, we consider the time interval $[t_j, t_{j+1}]$ instead of $[0, T]$, which is dictated by the necessity of rewriting (56) as (57) (see section 4.2). This, however, does not restrict the generality of the subsequent calculations.)

Introduce some notation. Let $Q = \Omega \times (t_j, t_{j+1})$ and let $W \subset L_2(Q)$ be a set of twice continuously differentiable functions $s(\mathbf{x}, t)$ defined in the closed

domain $\overline{Q} = \overline{\Omega} \times [t_j, t_{j+1}]$ and satisfying (74). Define on W the norm

$$||s||_H := \sqrt{\frac{1}{2}||s||_T^2 + \chi \sum_{l=1}^{n} \left\| \frac{\partial s}{\partial x_l} \right\|^2} \tag{76}$$

and the scalar product

$$(s_1, s_2)_H := \frac{1}{2}(s_1, s_2)_T + \chi \sum_{l=1}^{n} \left(\frac{\partial s_1}{\partial x_l}, \frac{\partial s_2}{\partial x_l} \right), \tag{77}$$

where $\chi > 0$, while

$$||s||_T := \sqrt{\int_\Omega |s(\mathbf{x}, t_{j+1})|^2 \, d\Omega},$$

$$(s_1, s_2)_T := \int_\Omega s_1(\mathbf{x}, t_{j+1}) s_2(\mathbf{x}, t_{j+1}) d\Omega,$$

and denote by $H(Q)$ the Hilbert space $(\overline{W}, || \cdot ||_H)$.

Definition 2. A function $s \in H(Q)$ is said to be a generalised solution to problem (73)-(75) if it satisfies the identity

$$\left(\frac{\partial s}{\partial t}, \psi \right) - (\nabla \cdot (D \nabla s), \psi) = (f_s, \psi) \tag{78}$$

for any $\psi \in H(Q)$ [24].

Theorem 2. For each $m \in \mathbb{N}$ the corresponding problem (73)-(75) is well-posed in the sense of existence, uniqueness, and stability of solution.

Proof. Since the proof is analogous to that given in section 5.1 for the advective subproblem, we shall only highlight its principal moments.

Existence. Let $a_j[s,\psi]$ $(j=1,2)$ denote the jth summand on the left-hand side of (78). Since the $a_j[s,\psi]$'s are linear bounded functionals with respect to s, it holds

$$a_j[s,\psi]=(s,v_j)_H \quad \forall s\in H(Q), \quad v_j\in H(Q), \quad j=1,2,$$

or, due to the correspondences $\psi\to v_j$, we have

$$a_j[s,\psi]=(s,L_j\psi)_H \quad \forall s\in H(Q), \quad j=1,2,$$

where $L_j: H(Q)\to H(Q)$. Similarly,

$$a_3[f_s,\psi]=(v_3,\psi)_H \quad \forall \psi\in H(Q), \quad v_3\in H(Q),$$

and hence

$$(s,L\psi)_H=(v_3,\psi)_H,$$

where $L=L_1+L_2$.

Lemma 2. The inverse operator $L^{-1}: R(H(Q))\to H(Q)$ exists and is bounded.

Proof. Let $q\in H(Q)$. Then

$$a_1[q,q]=\frac{1}{2}\,\|q\|_T^2\,,$$

$$a_2[q,q]\equiv-\left(\nabla\cdot(D\nabla q),q\right)=\int_{t_j}^{t_{j+1}}\int_\Omega D(\nabla q)^2 d\Omega dt\geq\mu\sum_{l=1}^{n}\left\|\frac{\partial q}{\partial x_l}\right\|^2,$$

where $0<\mu=\min\limits_{(\mathbf{x},t)\in\overline{Q}} D$. To obtain the last estimate we performed integration by parts and used the boundary condition (75). Choosing now χ in (77)

less than μ, we have

$$(q, Lq)_H = a_1[q,q] + a_2[q,q] \geq \frac{1}{2}\|q\|_T^2 + \mu \sum_{l=1}^{n} \left\| \frac{\partial q}{\partial x_l} \right\|^2 \geq (q,q)_H,$$

which proves the statement of the lemma. ■

Let L^{-1} be the inverse operator to L and let $\tilde{L}^{-1}$ be an extension of L^{-1} onto the whole $H(Q)$ such that $\tilde{L}^{-1}L = I$. The adjoint operator $(\tilde{L}^{-1})^*$ is unique and has the same norm as $\tilde{L}^{-1}$. Therefore, $\forall q = L\psi \in H(Q)$ the identity

$$(s,q)_H = (s, L\psi)_H = (v_3, \psi)_H = (v_3, \tilde{L}^{-1}q)_H = ((\tilde{L}^{-1})^* v_3, q)_H$$

holds iff $s = (\tilde{L}^{-1})^* v_3$. Existence is proved.

Uniqueness. Suppose, there are two different solutions, s_1 and s_2. Denote $\delta s = s_1 - s_2$; then it holds

$$\begin{aligned} \frac{\partial \delta s}{\partial t} - \nabla \cdot (D \nabla \delta s) = 0, \quad (\mathbf{x}, t) \in \Omega \times (t_j, t_{j+1}), \\ \delta s(\mathbf{x}, t)\Big|_{t=t_j} = 0, \\ \delta s(\mathbf{x}, t)\Big|_{\mathbf{x}\in\Gamma} = 0, \quad t_j < t \leq t_{j+1}. \end{aligned}$$

Substituting δs and $\psi = \delta s$ into (78), we find

$$\frac{1}{2}\int_\Omega \delta s^2(\mathbf{x}, t_{j+1}) d\Omega + \int_{t_j}^{t_{j+1}} \int_\Omega D(\nabla \delta s)^2 d\Omega dt = 0,$$

from where $\delta s = 0$. Uniqueness is proved.

Stability. Let s_1 and s_2 be two different solutions corresponding to the initial data $\{g_1, f_{s,1}\}$ and $\{g_2, f_{s,2}\}$, respectively. Denote $\delta s = s_1 - s_2$,

$\delta f_s = f_{s,1} - f_{s,2}$, and $\delta g = g_1 - g_2$. Then, under the assumption that $\psi = \delta s$, (78) has the form

$$\frac{1}{2}\|\delta s\|_T^2 + \mu \sum_{l=1}^{n} \left\| \frac{\partial \delta s}{\partial x_l} \right\|^2 \leq \frac{1}{2}\|\delta g\|_0^2 + \|\delta f_s\| \, \|\delta s\| \, ; \tag{79}$$

here $\|\delta g\|_0 := \sqrt{\int_\Omega |\delta g(\mathbf{x})|^2 d\Omega}$. Due to the boundedness of the left-hand side of (79) there are two constants, ς_1 and ς_2, such that

$$\frac{1}{2}\|\delta s\|_T^2 + \mu \sum_{l=1}^{n} \left\| \frac{\partial \delta s}{\partial x_l} \right\|^2 \leq \|\delta s\|_Q \left(\|\delta g\|_0 + \|\delta f_s\| \right),$$

where

$$\|\delta s\|_Q := \varsigma_1 \|\delta s\|_T + \varsigma_2 \sum_{l=1}^{n} \left\| \frac{\partial \delta s}{\partial x_l} \right\| . \tag{80}$$

Further, since

$$\|\delta s\|_T \leq \sqrt{2\|\delta s\|_Q \left(\|\delta g\|_0 + \|\delta f_s\| \right)},$$
$$\left\| \frac{\partial \delta s}{\partial x_l} \right\| \leq \sqrt{\frac{\|\delta s\|_Q \left(\|\delta g\|_0 + \|\delta f_s\| \right)}{\mu}}, \quad l = \overline{1, n},$$

taking into account (80) we find

$$\|\delta s\|_Q \leq K(\|\delta g\|_0 + \|\delta f_s\|), \quad K = \left(\sqrt{2}\varsigma_1 + \frac{n\varsigma_2}{\sqrt{\mu}} \right)^2 .$$

The theorem is proved. ∎

5.3 Physical adequacy of the ABCs

Let us show that apart from well-posedness of the resulting boundary value problems the constructed boundary conditions are adequate from the physical

point of view. Namely, it is known that the function z can be thought of as the density of a substance that propagates in a medium under the influence of the advective and diffusive processes [27, 31]. Therefore, it would be natural to expect that the boundary conditions imposed to simulate the solution on Γ will always produce non-negative values for z.

For the advective subproblem the non-negativity of the solution on the boundary is evident: Because the function g may (from the physical standpoint) only be positive-valued (or zero-valued as well), the solution $z(\mathbf{x}, t)\Big|_{\mathbf{x}\in\Gamma^-} = g(\mathbf{x} - \mathbf{U}t)\Big|_{\mathbf{x}\in\Gamma^-}$ is certainly positive (or zero).

To show the non-negativity in the diffusive case, for simplicity we shall assume $n = 1$, as well as consider the spatial derivatives in (57) to be discretised. For instance, if $m = 1$ then it holds

$$z_0^{[j+1]} = z_0^{[j]} + D\tau_j \frac{z_{-1}^{[j]} - 2z_0^{[j]} + z_{+1}^{[j]}}{\Delta x_1^2}. \tag{81}$$

Since (81) is a finite difference scheme, we shall involve the von Neumann analysis of stability [32] and the Babenko-Gelfand criterion [12]. Namely, it is known that in order for (81) to be stable it is sufficient that $4D\tau_j \leq \Delta x_1^2$. Hence, we obtain

$$z_0^{[j+1]} = z_0^{[j]} + D\tau_j \frac{\lambda^j e^{-i\alpha\Delta x_1} - 2\lambda^j + \lambda^j e^{+i\alpha\Delta x_1}}{\Delta x_1^2} = \cdots =$$
$$= z_0^{[j]} \left(1 + 2D\tau_j \frac{\cosh i\alpha\Delta x_1 - 1}{\Delta x_1^2}\right), \quad \lambda \in \mathbb{C}, \quad \alpha \in \mathbb{R}, \quad i^2 = -1,$$

and estimating from below $\cosh i\alpha\Delta x_1 \geq -1$, come to

$$z_0^{[j+1]} \geq z_0^{[j]} \left(1 + 2 \cdot \frac{1}{4} \cdot -2\right) = 0.$$

Generalisation of this result for arbitrary $n > 1$ and $m > 1$ is trivial.

Consequently, aside from the properties of exactness, locality, and geometrical flexibility, the constructed artificial boundary conditions are both mathematically (due to the theorems proved) and physically (due to the last remark) correct.

5.4 Computational efficiency of the time splitting

As a concluding remark, let us observe that within the framework of derivation of ABCs the time splitting allows to significantly reduce the size of computational domain in the outflow direction when solving various fluid dynamics problems, and thus to decrease the computational cost of solution, especially for multidimensional equations.

Indeed, consider the classical problem of the flow round a body (Figure 1). (Here, since we are dealing with the advection-diffusion equation rather than with a Navier-Stokes system, we assume the velocity field $\mathbf{U}$ to be known.) It is known [25] that the presence of the obstacle (G in the figure) creates behind it a so-called laminar wake. Therefore, beyond the body the field $\mathbf{U}$ can be assumed to be constant almost everywhere except the area of the wake. Consequently, the construction of an outflow ABC for the whole advection-diffusion equation requires one to place the outflow boundary sufficiently far from the obstacle ($\Gamma^+ = \Gamma_1^+$ in Figure 1), in order for the supposition about the constancy of the velocity field to be valid and for some or other analytical methods (for deriving the ABC) to be applicable. Obviously, however, that

this move will increase the size of the domain Ω and therefore make the solution rather expensive from the computational standpoint. Nevertheless, in case of using the time splitting we are avoided from constructing outflow ABCs for the advective problem, just because on Γ^+ the solution is simply computed by an appropriate interior finite difference scheme. Hence, the presence of the laminar wake does not imply the boundary to be placed far from the body, and so we are able to substantially decrease the size of Ω in the outflow direction ($\Gamma^+ = \Gamma_2^+$ in the figure) and thereby reduce the computational cost of solution.

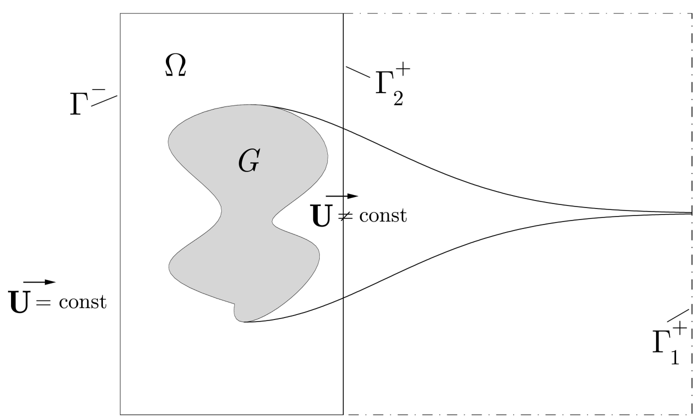

Figure 1: Due to the time splitting we reduce the size of the domain Ω in the outflow direction and thereby substantially decrease the computational cost of solution

6 CONCLUSIONS

We have investigated the method of operator splitting as an approach to the construction of artificial boundary conditions for numerical solution to partial differential equations in unbounded domains. A general methodology was developed for the linear advection-diffusion-reaction equation. The derived ABCs are local both in time and in space, that is implementable from the computational standpoint; besides, they appear to be exact, which allows highly accurate calculating the solution at the boundary points and in the interior; furthermore, the ABCs are uncritical to the shape of artificial boundary, which is extremely important for a huge number of practical advection-diffusion(-reaction) problems. The resulting boundary value problems are well-posed as well as adequate from the physical point of view. We expect the methodology admits generalisations to other classes of differential problems where the question of artificial boundary conditions arises.

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Convergence of Fourier series in Hölder norms

Miguel Antonio Jiménez-Pozo[1]

Abstract *For 1< $p<\infty$ and $\alpha>0$, we prove the convergence in Hölder norm of the Fourier series of $p-$integrable Hölder-Zygmund functions of order α and present an estimate of the degree of this convergence.*

Keywords and phrases: Hölder approximation, Lipschitz function, Zygmund class, 0-equicontinuous set, Fourier series.

Mathematics Subject Classification: 26A16, 41A65, 42A20

1. Introduction

Denote by $L_{2\pi}^p$, $1\leq p<\infty$ (respectively by $L_{2\pi}^{\infty}:=C_{2\pi}$), the Banach spaces of $2\pi-$periodic $p-$integrable (respectively continuous) functions with Lebesgue measure $dx/2\pi$.

Let $f\in L_{2\pi}^p, 1\leq p\leq\infty$. For every $t>0$, define the modulus of continuity

$$\omega\left(f,t\right)_p:=\sup\left\{\left\|\Delta_s f\right\|_p:0<s\leq t\right\},$$

where $\Delta_s f:=[f]_s-f$, $[f]_s\left(x\right):=f\left(x+s\right)$ and $\left\|f\right\|_p:=(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(t\right)\right|^p dt)^{1/p}$ is the $p-$norm if $1\leq p<\infty$ or $\left\|f\right\|_{\infty}:=\sup\left\{\left|f\left(x\right)\right|:x\in[0,2\pi]\right\}$ is the sup-norm.

Let $0<\alpha<1$. For every $\delta>0$, define

$$\theta_\alpha\left(f,\delta\right)_p:=\sup\left\{\frac{\omega\left(f,t\right)_p}{t^\alpha}:0<t\leq\delta\right\}$$

and set

$$\theta_\alpha\left(f\right)_p:=\sup\left\{\theta_\alpha\left(f,\delta\right)_p:\delta>0\right\}.$$

We say that $f\in L_{2\pi}^p$, $1\leq p\leq\infty$, satisfies a Hölder (or Lipschitz) condition of order α, if $\theta_\alpha^p\left(f\right)<\infty$ and denote the collection of all those functions by Lip_α^p. It is known that Lip_α^p becomes a Banach space under the Hölder norm

$$\left\|f\right\|_{p,\alpha}:=\left\|f\right\|_p+\theta_\alpha\left(f\right)_p. \tag{1.1}$$

Every differentiable function f and in particular every trigonometric polynomial, satisfies the property of annulation

$$\theta_\alpha\left(f,\delta\right)_p\longrightarrow 0 \text{ as } \delta\longrightarrow 0,\ 1\leq p\leq\infty. \tag{1.2}$$

Let $f\in Lip_\alpha^p$, $0<\alpha<1$, $1\leq p\leq\infty$ and denote by

$$S_n\left(f,x\right):=\sum_{-n}^{n}\left\langle f,k\right\rangle\exp\left(ikx\right),$$

the partial sums of the Fourier series of f.

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Since $S_n(f)$ are trigonometric polynomials, the function f must satisfy property (1.2) if the convergence $\|S_n(f) - f\|_{p,\alpha} \to 0$ is expected. Consequently, we introduce the Banach subspaces lip^p_α, by means of those functions of Lip^p_α that satisfy this property of annulation. Mirkil theorem [10] states that trigonometric polynomials are dense in lip^p_α for all $0 < \alpha < 1$ and all $1 \le p \le \infty$. We refer to the basic references [1], [2], [3] and [5] for details and complements.

Now we concentrate our efforts in the study of the convergence of Fourier series with Hölder norm. A basic reference here is [12]. It is known that neither in lip^∞_α nor in lip^1_α, the partial sums of the Fourier series are approximation processes. This motivates other methods of summation of the Fourier series in Hölder norms (See references [3], [9], or more recently [11]). For $1 < p < \infty$, the situation is quite different. In fact, we are going to present not only a simple proof of the convergence of Fourier series in Hölder norm in this case but also an estimate of the degree of convergence in terms of $\theta_\alpha(.,.)_p$, defined below in a general way and used here as a modulus of smoothness.

First, suppose we extend the definitions above in a natural way for $\alpha \ge 1$. Then, for $\alpha = 1$, we find the classical classes Lip^p_1 of Lipschitz functions. In Lip^p_1, the property of annulation (1.2) does not hold in general for polynomials because $([f]_t - f)\diagup t$ approaches the derivative $f\acute{}$ as $t \to 0$. Indeed, combining Shilov and Hardy-Littlewood theorems (see [3]), the closure of the trigonometric polynomials in Hölder norm in Lip^∞_1 is $C^1_{2\pi}$, in $Lip^p_1, 1 < p < \infty$ are the linear spaces of antiderivatives of L^p functions and in Lip^1_1 is the space of functions of bounded variation. On the other hand, for $\alpha > 1$ the spaces $Lip^p_{2\pi}$ are reduced to constant functions. In view of this situation we will not use the natural extension of the concepts. Instead, we introduce the Zygmund definition as follows:

Define by induction $\Delta^r_s f := \Delta^1_s\left(\Delta^{r-1}_s f\right), r = 2, 3, ...$ where $\Delta^1_s f := \Delta_s f$ and the $r-$modulus of smoothness by

$$\omega_r(f,t)_p := \sup\left\{\|\Delta^r_s f\|_p : 0 < s \le t\right\}, \tag{1.3}$$

where $\omega_1(f,t)_p := \omega(f,t)_p$. Now, if $\alpha > 0$ is given, we choose $r := r(\alpha) := [\alpha] + 1$ (i. e. the smallest integer r, such that $\alpha < r$) and set in a general way

$$\theta_\alpha(f,\delta)_p := \sup\left\{\frac{\omega_r(f,t)_p}{t^\alpha} : 0 < t \le \delta\right\}. \tag{1.4}$$

Now we can define $\theta_\alpha(f)_p$ as before and say that $f \in L^p_{2\pi}$, $1 \le p \le \infty$, satisfies the Zygmund-Hölder condition of order α, if $\theta^p_\alpha(f) < \infty$. We still denote the collection of all those functions by Lip^p_α and use (1.1) to define the norms in these spaces. It is known that Lip^p_α is a Banach space. Finally we extend the definition of the Banach subspaces lip^p_α, by means of those functions of Lip^p_α that satisfy the property of annulation (1.2). Relations between classical spaces, the Zygmund-Hölder spaces, Sobolev spaces, Besov spaces, etc., are well known and described in the references given here. Now we are ready to state the result of this note.

2. The main result

THEOREM 1. *For every $f \in lip^p_\alpha$, $\alpha > 0$ and $1 < p < \infty$, it happens that $\|S_n(f) - f\|_{p,\alpha} \to 0$ as $n \to \infty$. Moreover, if*

$$\|S_n(f) - f\|_p = \mathbf{O}\left(\omega_{r(\alpha)}(f, \psi(n))_p\right), \tag{2.1}$$

where $\psi(n)$ is a sequence that decreases to zero, then

$$\left\| S_n(f) - f \right\|_{p,\alpha} = \mathbf{O}\ \left(\theta_\alpha(f, \psi(n))_p \right). \tag{2.2}$$

PROOF. Fix the parameters α and p.

First, observe that using (1.3), we can substitute the traditional definition (1.4) by

$$\theta_\alpha(f,\delta)_p := \sup\left\{ \frac{\|\Delta_t^r f\|_p}{t^\alpha} : 0 < t \le \delta \right\}. \tag{2.3}$$

Also observe that

$$[S_n(f)]_t(x) = (S_n([f]_t))(x).$$

Thus, using the linearity of S_n we get that $\Delta_t^1(S_n f) = S_n\left(\Delta_t^1 f\right)$ and by induction

$$\Delta_t^r(S_n f) = S_n(\Delta_t^r f),\ r = 1, 2, ... \tag{2.4}$$

It is well known (see [8]) that

$$\forall f \in L_{2\pi,}^p\ \|S_n(f) - f\|_p \to 0 \text{ as } n \to \infty. \tag{2.5}$$

In particular, from the uniform boundedness principle, the sequence of operator norms $(\|S_n\|)$ between $L_{2\pi}^p$ spaces is uniformly bounded by a constant $M > 0$ which depends on p but on nothing else. Keep this in mind.

We need to prove that

$$\forall f \in lip_\alpha^p,\ \theta_\alpha(S_n f - f)_p \to 0 \text{ as } n \to \infty.$$

Then it is sufficient to prove that

$$\forall f \in lip_\alpha^p,\ \theta_\alpha(S_m(f) - S_n(f))_p \to 0 \text{ as } m, n \to \infty. \tag{2.6}$$

In fact, from (1.1), (2.5) and (2.6), it follows that $(S_n(f))$ is a Cauchy sequence in the complete space lip_α^p and once again by (2.5) that its limit must be f.

For every $\delta > 0$ we can write

$$\begin{aligned} \theta_\alpha(S_m(f) - S_n(f))_p \le&\\ &\theta_\alpha(S_m(f), \delta)_p + \theta_\alpha(S_n(f), \delta)_p + \\ &\sup\left\{ \frac{\|\Delta_t^r(S_m(f) - S_n(f))\|_p}{t^\alpha} : t \ge \delta \right\}, \end{aligned} \tag{2.7}$$

where the new definition given in (2.3) was employed.

Using (2.4) and the boundedness of the sequence $(\|S_n\|)$, we get

$$\begin{aligned} \theta_\alpha(S_k(f), \delta)_p &= \sup\left\{ \frac{\|\Delta_t^r(S_k f)\|_p}{t^\alpha} : 0 < t \le \delta \right\} = \sup\left\{ \frac{\|S_k(\Delta_t^r f)\|_p}{t^\alpha} : 0 < t \le \delta \right\} \\ &\le M \sup\left\{ \frac{\|\Delta_t^r f\|_p}{t^\alpha} : 0 < t \le \delta \right\} = M\theta_\alpha(f,\delta)_p, \end{aligned}$$

and then from (2.7)

$$\theta_\alpha(S_m(f) - S_n(f))_p \le 2M\theta_\alpha(f,\delta)_p + \sup\left\{ \frac{\|\Delta_t^r(S_m(f) - S_n(f))\|_p}{t^\alpha} : t \ge \delta \right\} \tag{2.8}$$

Also, since $\|.\|_p$ is translation invariant,

$$\|\Delta_t^r(S_m(f) - S_n(f))\|_p \le C\,\|S_m(f) - S_n(f)\|_p,$$

where the constant C depends on r (which depends on α) but on nothing else.

By substituting this inequality into (2.8), we get

$$\theta_\alpha \left(S_m\left(f\right)-S_n\left(f\right)\right)_p \leq 2M\theta_\alpha \left(f,\delta\right)_p+\delta^{-\alpha}C\left\|S_m\left(f\right)-S_n\left(f\right)\right\|_p . \tag{2.9}$$

To get (2.6) and finish the proof of the qualitative part of the theorem, first take δ small and further m and n large enough in (2.9).

Once we know that $S_n\left(f\right)$ converges to f we proceed to estimate the degree of convergence as follows. For any fixed n, δ and t, first take limit in (2.9) with respect to m and after that set $\delta=\psi\left(n\right)$ to obtain

$$\left\|S_n\left(f\right)-f\right\|_{p,\alpha} \leq \left\|S_m\left(f\right)-f\right\|_p+2M\theta_\alpha\left(f,\psi\left(n\right)\right)_p+C\psi\left(n\right)^{-\alpha}\ \ \left\|S_n\left(f\right)-f\right\|_p .$$

By substituting the hypothesis (2.1) into this last inequality we easily obtain our claim (2.2). $\square$

Remark 1. *We could prove this theorem with the help of the tauberian theorem given by the author in* [6] *and* [7]. *Anyway we should accomplish many of the steps given above and to place and identify different hypothesis as well. Then, not so much simplification (if any) could be obtained. Instead, we catch the idea of that tauberian theorem and give here a self-contained proof of the result in his context.*

Remark 2. *Of course, replacing S_n by comfortable sequences of linear operators arising from convolution processes of summation of Fourier series with kernels such as the classical de la Vallée-Poussin, Féjer and Jackson among others or new ones such as Chandra* [4], *we can estimate their degrees of approximation in Hölder norms with the same method employed in the proof of the theorem and to include other cases such as $p=1$.*

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Solution of Parameter-Varying Linear Matrix Inequalities in Toeplitz Form

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Abstract

In this paper the necessary and sufficient conditions are given for the solution of a system of parameter varying linear inequalities of the form $\mathbf{A}(t)\,\mathbf{x} \geq \mathbf{b}(t)$ for all $t \in T$, where T is an arbitrary set, $\mathbf{x}$ is the unknown vector, $\mathbf{A}(t)$ is a known triangular Toeplitz matrix and $\mathbf{b}(t)$ is a known vector. For every $t \in T$ the corresponding inequality defines a polyhedron, in which the solution should exist. The solution of the linear system is the intersection of the corresponding polyhedrons for every $t \in T$. A general modular decomposition method has been developed, which is based on the successive reduction of the initial system of inequalities by reducing iteratively the number of variables and by considering an equivalent system of inequalities.

Keywords: Linear matrix inequalities, parameter varying systems, constrained optimization, polyhedron, robust control theory, Toeplitz matrices.

AMS classification: 52B99, 47A50, 15A39, 15A45, 51M20, 05A20.

1 Introduction

A wide variety of problems arising in system and control theory can be reduced to constrained optimization problems, having as design constraints a simple reformulation in terms of linear matrix inequalities [1],[5]. Parameter varying Linear Matrix Inequalities (LMIs) have been proved to be a powerful tool, having important applications in a vast variety of systems and control theory problems including robustness analysis, robust control synthesis, stochastic control and identification [3],[2], synthesis of dynamic output feedback controllers [7], analysis and synthesis of control systems [4], error and sensitivity analysis, problems encountered in filtering, estimation, etc. Specifically, LMIs appear in the solution of continuous and discrete-time H_∞ control problems, in finding solvability conditions for regular and singular problems, in parameterization of H_∞ and H_2 suboptimal controllers, including reduced-order controllers [6], in finding explicit controller formulas of the H_∞ synthesis [1],[5], as well as in multiobjective synthesis and

in linear parameter-varying synthesis.

LMI techniques offer the advantage of operational simplicity in contrast with the classical approaches, which necessitate the cumbersome material of Riccati equations [1]. Using LMIs, a small number of concepts and principles are sufficient to develop tools, which can then be used in practice. Also, the LMI techniques are effective numerical tools exploiting a branch of convex programming. Many LMI control methods make use of Lyapunov variables and possibly additional variables, often called scalings or multipliers, which in some sense translate how ideal behaviors are altered by uncertainties or perturbations.

Another application of LMIs is the domain of graphical manipulation in dynamic environments, where the types of interactive controls are restricted by reducing the problem of graphical manipulation to a constrained optimization problem, dictating how a user configures a set of graphical objects to achieve the desired goals. Thus, the possible configurations of the objects are represented by the object's state vector having a set of real-valued parameters and the graphical interaction problem is reduced to a problem of resolving the corresponding system of LMIs [8].

In this paper we provide necessary and sufficient conditions for the existence of the solution of the system of inequalities $\mathbf{A}(t)\,\mathbf{x} \geq \mathbf{b}(t)\,, \forall t \in T$ and restrictions of this solution, if such exists, in the general case, where T may be an infinite, or even a super countable set. Specifically, t is a variable within an arbitrary set T, which may represent the domain of external disturbances or parameter variations of a system in the most general form, $\mathbf{x} \in \mathrm{R}^N$ is the unknown vector, $\mathbf{A}(t) \in \mathrm{R}^{N\times N}$ is a given triangular Toeplitz matrix dependent on t and $\mathbf{b}(t) \in \mathrm{R}^N$ is a given vector of parameters dependent on t. A Toeplitz matrix $\mathbf{A} = [a_{ij}] \in \mathrm{R}^{N\times N}$is a highly structured matrix, where $a_{i+1,j+1} = a_{i,j}$, for each appropriate $i, j \in \{1, 2, ..., N\}$, containing at most $2N - 1$ different element values. The use of a triangular Toeplitz matrix finds many applications in control theory and signal processing, since every element of the vector $\mathbf{A}(t)\,\mathbf{x}$ is a discrete-time convolution between the sequence of the functions in $\mathbf{A}(t)$ and the sequence in $\mathbf{x}$ and so the inequality $\mathbf{A}(t)\,\mathbf{x} \geq \mathbf{b}(t)$ represents a convolution that is greater than or equal to a given function, at every moment. Also a Toeplitz Matrix is the covariance matrix of a weak stationary stochastic process.

The case, where $T = \{t_0\}$ is an one-element set, can be solved with various methods, like the ellipsoid-algorithm [10]. Then, the case of a finite set T is a generalization of the latter case, in the sense that one can consider $|T|$ times the special problem on an one-element set. On the other hand, the most general cases, where the set T is infinite and in particular where T is super countable (for example when $T = \mathrm{R}^k, k \in \mathrm{N}$), are of major importance and are considered here. Although the system of equations $\mathbf{A}(t)\,\mathbf{x} = \mathbf{b}(t)\,, \forall t \in T$ has numerous methods of solutions, there is no available algorithm allowing computing the solutions of a system of inequalities $\mathbf{A}(t)\,\mathbf{x} \geq \mathbf{b}(t)\,, \forall t \in T$ in the general case of infinite T [11],[9].

The underlying idea in the present paper for the solution of the LMIs $\mathbf{A}(t)\,\mathbf{x} \geq \mathbf{b}(t)\,, \forall t \in T$ is the General Modular (GM) decomposition of the involved inequalities into simpler inequalities, considering

the cases where each element $a_i(t)$ of $\mathbf{A}(t) \in \mathrm{R}^{N\times N}$ takes zero, positive or negative values. This is possible, since a given inequality is reduced to different simpler inequalities for different ranges of $t \in T$.
Following this reasoning, in Section 2 an arbitrary inequality with $k = 1$ variable is decomposed into three inequalities, the first of them including only known coefficients, including no variable and the other two expressing explicitly the upper and the lower range respectively of this one variable. Also an arbitrary inequality, including $k \geq 2$ variables is decomposed into four inequalities, each one including $k-1$ variables, using the GM decomposition. In both decompositions we derive a set of inequalities, which have a solution, if and only if the initial inequality has a solution. In Section 3 the decompositions described in Section 2 are applied successively $k-1$ times to an arbitrary inequality with $k \geq 2$ variables, thus arriving at a set of inequalities including exactly one of the k variables. Each of these inequalities of one variable is further decomposed into three inequalities. The main results of the present contribution are (a) the necessary and sufficient conditions of the existence of a solution $\mathbf{x}$ of the system and (b) the restrictions of the solution, which are expressed in the form of a hypercube, i.e. the upper and lower bound for each unknown variable x_r, $1 \leq r \leq N$, in the case where such a solution exists, which are derived in Section 4 in analytic form.

2 Decomposition of Inequalities

In this section, the decomposition of a given inequality for $t \in T$ into simpler inequalities that hold for t belonging in subsets of T, so that the polynomials $a_i(t)$, $i = 1, 2, ..., N$ take zero, positive and negative values, are described. These sets constitute a partition of T. Here T is arbitrary and plays the role of an external parameter-set, which may represent an one- or multidimensional variable (vector) that is dependent on time and other parameters. This partition of T is given in Definition 1.

Definition 1. Let $a_i(t)$, $\forall i \in \{1, 2, ..., k\}$, be a certain sequence of functions dependent on $t \in T$, for an arbitrary set T. Then, we define for each $i \in \{1, 2, ..., k\}$ the partition sets of T:

$$S_i^1 = \{t \in T : a_i(t) = 0\}, \quad S_i^2 = \{t \in T : a_i(t) > 0\}, \quad S_i^3 = \{t \in T : a_i(t) < 0\}.$$

The underlying idea is that the partition of the set T into three subsets S_i^1, S_i^2, S_i^3 leads to inequalities having the restriction that the functions $a_i(t)$, $\forall i \in \{1, 2, ..., N\}$ are zero-, positive- or negative-valued respectively, where $a_i(t)$ are the elements of the Toeplitz matrices $\mathbf{A}(t) \in \mathrm{R}^{N\times N}$ appearing in the LMIs $\mathbf{A}(t)\mathbf{x} \geq \mathbf{b}(t), \forall t \in T$.

Based on the above approach, in the rest of this Section the following results are presented:

- Lemma 1 describes the Special Decomposition of an arbitrary inequality in $k = 1$ variable, into three equivalent inequalities, the first of them having only known quantities with no variables and the other two expressing explicitly the upper and lower bound for this one variable, in order to

satisfy the initial inequality.

- Theorem 1 provides the necessary and sufficient conditions for the existence of a solution of an arbitrary inequality in $k \geq 2$ variables, in the form of two inequalities each one of them including $k-1$ variables and

- Theorem 2 describes the General Modular (GM) Decomposition of an arbitrary inequality in $k \geq 2$ variables into four equivalent inequalities, each one of them including $k-1$ variables.

Lemma 1 (Special Decomposition of inequalities in one variable)**.** *There exists $x_1 \in \mathrm{R}$, so that:*

$$a_1(t)\, x_1 \geq b(t)\,,\ \forall t \in T \tag{1}$$

if and only if exists $x_1 \in \mathrm{R}$, such that the following inequalities hold:

$$0 \geq b(t)\,,\ \forall t \in S_1^1, \tag{2}$$

$$\frac{b(t)}{a_1(t)} \leq x_1,\ \forall t \in S_1^2, \tag{3}$$

$$x_1 \leq \frac{b(t)}{a_1(t)},\ \forall t \in S_1^3, \tag{4}$$

The above three inequalities (2), (3) and (4) constitute the Decomposition of (1).

Proof. (a). Necessary condition. Suppose that exists $x_1 \in \mathrm{R}$, so that (1) holds. Then (2) holds since $a_1(t) = 0, \forall t \in S_1^1$. Also for $t \in S_1^2$ and $t \in S_1^3$, the relations (3) and (4) hold respectively. Therefore x_1 satisfies (2)-(4) depending on the range of t in the sets S_1^1, S_1^2, S_1^3 and the necessity part has been proved.

(b). Sufficient condition. Conversely, suppose that $\exists x_1 \in \mathrm{R}$, so that (2)-(4) hold. Then

- for $a_1(t) = 0,\ \forall t \in S_1^1$and from (2) it results that $a_1(t)\, x_1 = 0 \geq b(t)\,,\ \forall t \in S_1^1$

- for $a_1(t) > 0,\ \forall t \in S_1^2$and from (3) it results that $a_1(t)\, x_1 \geq b(t)\,,\ \forall t \in S_1^2$

- for $a_1(t) < 0,\ \forall t \in S_1^3$and multiplying (4) with the negative quantity $a_1(t) = -|a_1(t)|$, it results that $a_1(t)\, x_1 \geq b(t)\,,\ \forall t \in S_1^3$.

Thus, it holds $a_1(t)\, x_1 \geq b(t)\,,\ \forall t \in T = S_1^1 \cup S_1^2 \cup S_1^3$, from which the sufficient part of Lemma 1 is concluded. □

Theorem 1. *Suppose we have the inequality:*

$$\sum_{i=1}^{k} a_{k-i+1}(t)\, x_i \geq b(t)\,, \forall t \in T, \quad k \geq 2 \tag{5}$$

where $a_i(t)\,,\ i \in \{1, 2, ..., k\}$ and $b(t)$are varying coefficients dependent on t and $x_i,\ i \in \{1, 2, ..., k\}$ are unknown real variables independent of t. There exists a vector $\mathbf{x} = [x_1, x_2, ..., x_k]^T \in \mathrm{R}^k$ satisfying (5), if

and only if there exists a vector $\mathbf{x}' = [x'_1, x'_2, ..., x'_k]^T \in \mathrm{R}^k$ *satisfying the conditions:*

$$\sum_{i=1}^{k-1} a_{k-i+1}(t)\, x'_i \geq b(t), \forall t \in S_1^1, \tag{6}$$

$$\sum_{i=1}^{k-1} \left[\frac{a_{k-i+1}(t_2)}{|a_1(t_2)|} + \frac{a_{k-i+1}(t_3)}{|a_1(t_3)|} \right] x'_i \geq \left[\frac{b(t_2)}{|a_1(t_2)|} + \frac{b(t_3)}{|a_1(t_3)|} \right], \forall (t_2, t_3) \in S_1^2 \times S_1^3. \tag{7}$$

Proof. (a). Necessary condition. Suppose that there exist some $\mathbf{x} = [x_1, x_2, ..., x_k]^T \in \mathrm{R}^k$, such that (5) holds. For $a_1(t) = 0,\ \forall t \in S_1^1$ it is seen from (5) that (6) holds, while for $a_1(t) > 0,\ \forall t \in S_1^2$ and $a_1(t) < 0,\ \forall t \in S_1^3$, we have respectively:

$$x_k \geq \frac{[b - a_k x_1 - ... - a_2 x_{k-1}](t)}{a_1(t)},\ \forall t \in S_1^2$$

and

$$x_k \leq \frac{[b - a_k x_1 - ... - a_2 x_{k-1}](t)}{a_1(t)},\ \forall t \in S_1^3,$$

which are satisfied only when:

$$\frac{[b - a_k x_1 - ... - a_2 x_{k-1}](t_2)}{a_1(t_2)} \leq \frac{[b - a_k x_1 - ... - a_2 x_{k-1}](t_3)}{a_1(t_3)},\ \forall (t_2, t_3) \in S_1^2 \times S_1^3. \tag{8}$$

The use of the Cartesian product in (8) dictates the use of the auxiliary independent variables $t_2 \in S_1^2$, $t_3 \in S_1^3$. Inequality (8) is equivalent to (7) for $\mathbf{x}' = \mathbf{x}$, since $a_1(t_2) = |a_1(t_2)|,\ \forall t_2 \in S_1^2$ and $a_1(t_3) = -|a_1(t_3)|,\ \forall t_3 \in S_1^3$. Therefore both conditions (6) and (7) are satisfied for $\mathbf{x}' = \mathbf{x}$ and the necessity part has been proved.

(b). Sufficient condition. Conversely, suppose that there exists some $\mathbf{x} = [x_1, x_2, ..., x_k]^T \in \mathrm{R}^k$, so that (6) and (7) hold. It will be shown that exists a vector $\mathbf{x}' = [x'_1, x'_2, ..., x'_k]^T \in \mathrm{R}^k$, in general different from $\mathbf{x}$, for which (5) also holds.

Inequality (7) is equivalent to (8) (when substituting $\mathbf{x}'$ by $\mathbf{x}$), which may be written as:

$$\exists c \in \mathrm{R}: \ \frac{[b - a_k x_1 - ... - a_2 x_{k-1}](t_2)}{a_1(t_2)} \leq c \leq \frac{[b - a_k x_1 - ... - a_2 x_{k-1}](t_3)}{a_1(t_3)},\ \forall (t_2, t_3) \in S_1^2 \times S_1^3. \tag{9}$$

Multiplying the left and right part of the inequalities in (9) with $a_1(t_2) > 0$ and $a_1(t_3) < 0$ respectively and summarizing the results, it results that (9) is equivalent to the inequality

$$a_k(t)\, x_1 + ... + a_2(t)\, x_{k-1} + a_1(t)\, c \geq b(t),\ \forall t \in S_1^2 \cup S_1^3. \tag{10}$$

Since $a_1(t) = 0,\ \forall t \in S_1^1$, we obtain from (6) (substituting also $\mathbf{x}'$ by $\mathbf{x}$):

$$\sum_{i=1}^{k-1} a_{k-i+1}(t)\, x_i = a_k(t)\, x_1 + a_{k-1}(t)\, x_2 + ... + a_2(t)\, x_{k-1} + a_1(t)\, c \geq b(t), \forall t \in S_1^1. \tag{11}$$

Now, from (10) and (11) it follows that:

$$a_k(t)\,x_1 + \ldots + a_2(t)\,x_{k-1} + a_1(t)\,c \geq b(t)\,,\ \forall t \in T = S_1^1 \cup S_1^2 \cup S_1^3, \tag{12}$$

from which we can see that there exists a solution $\mathbf{x}' = [x_1, x_2, \ldots, x_{k-1}, c]^T \in \mathrm{R}^k$ for (5). This proves the sufficient part of Theorem 1. $\square$

Theorem 1 gives the necessary and sufficient conditions (6) and (7) for the existence of solutions of (5). Using this equivalence, where only one variable is eliminated, we loose information about the conditions that this variable should satisfy. Indeed, in (6) and (7) the variable x_k has been removed and the information about the range of the values that x_k may take in an eventual solution of (5) is lost.

The idea which is used in order to recover the information about x_k is the additional elimination of another variable, say of x_{k-1}, so that a second pair of inequalities similar to (6) and (7) are derived, which have a solution if and only if (5) has a solution. Thus, by the elimination of two variables x_k and x_{k-1}, we arrive at the following Theorem 2, which describes the General Modular (GM) Decomposition of the initial inequality (5) into a set of four equivalent inequalities, each one of them including $k-1$ variables, without losing information about the range of the variables in the solution.

Theorem 2 (General Modular (GM) Decomposition of (5))**.** *The inequality (5) can be decomposed equivalently into the following four inequalities:*

$$\sum_{i=1}^{k-1} a_{k-i+1}(t)\,x_i \geq b(t)\,,\ \forall\, t \in S_1^1, \tag{13}$$

$$\sum_{i=1}^{k-1} a_{k-i+1,1}\left(\bar{t}\right) x_i \geq b_1\left(\bar{t}\right),\ \forall\, \bar{t} = (t_2, t_3) \in S_1^2 \times S_1^3 \tag{14}$$

$$\sum_{\substack{i=1 \\ i \neq k-1}}^{k} a_{k-i+1}(t)\,x_i \geq b(t)\,,\ \forall\, t \in S_2^1, \tag{15}$$

$$\sum_{\substack{i=1 \\ i \neq k-1}}^{k} a_{k-i+1,2}\left(\bar{t}\right) x_i \geq b_2\left(\bar{t}\right),\ \forall\, \bar{t} = (t_2, t_3) \in S_2^2 \times S_2^3, \tag{16}$$

where:

$$a_{k-i+1,1}\left(\bar{t}\right) = \frac{a_{k-i+1}(t_2)}{|a_1(t_2)|} + \frac{a_{k-i+1}(t_3)}{|a_1(t_3)|},\ b_1\left(\bar{t}\right) = \frac{b(t_2)}{|a_1(t_2)|} + \frac{b(t_3)}{|a_1(t_3)|},\ \forall\, \bar{t} = (t_2, t_3) \in S_1^2 \times S_1^3, \tag{17}$$

$$a_{k-i+1,2}\left(\bar{t}\right) = \frac{a_{k-i+1}(t_2)}{|a_2(t_2)|} + \frac{a_{k-i+1}(t_3)}{|a_2(t_3)|},\ b_2\left(\bar{t}\right) = \frac{b(t_2)}{|a_2(t_2)|} + \frac{b(t_3)}{|a_2(t_3)|},\ \forall\, \bar{t} = (t_2, t_3) \in S_2^2 \times S_2^3. \tag{18}$$

This set of inequalities (13)-(16) has a solution if and only if the inequality (5) has a solution.

Proof. It follows directly from Theorem 1 that each set of inequalities (13), (14) and (15), (16) constitute a set of equivalent conditions for the solution of (5). Moreover, the use of both pairs of inequalities guarantees that no information about the range of the variables is lost. □

In order to simplify the solution of the problem, we further decompose iteratively the initial inequality (5) into inequalities that contain a smaller number of variables, according to the GM decomposition. The technical advantage of the GM decomposition is, that only the sets S_1^1, S_1^2, S_1^3 and S_2^1, S_2^2, S_2^3, in which the coefficients of the two last variables x_{k-1} and x_k are respectively null, positive or negative, are used. This decomposition constitutes the substructure for the determination of the complete set of the conditions that the solutions of (5) should satisfy. These conditions determine the hypercube, where the solutions lie.

3 Reduction of an arbitrary inequality

In this section the initial inequality of the form (5) is reduced to a number of equivalent simpler inequalities that will be called “implicit” inequalities. This reduction is presented in Theorem 3 and is achieved in two steps:

- *Step 1.* Application of the GM decomposition successively $(k-1)$ times to an arbitrary inequality on $k \geq 2$ variables, leading at the end to a set of inequalities, each one of them containing implicitly one variable.
- *Step 2.* Application of the Special Decomposition described in Lemma 1 to each one of the inequalities resulted from Step 2, leading to a set of inequalities equivalent to the initial inequality, each one of them containing either only known quantities with no variables or explicitly only one variable.

At the 0^{th} decomposition-level consider that there is the inequality (5), while at the 1^{st} decomposition-level the inequalities (13)-(16) appear. Continuing in this way and applying iteratively the GM decomposition, we arrive at the $j^{th},\ j \in \{1, 2, ..., k-1\}$ decomposition-level.

It is seen from (14) and (16) that the coefficients of the variables after the application of the GM decomposition are functions of the coefficients $a_i\,(t)\,,\ i \in \{1, 2, ..., k\}$ of the given inequality (5), while in (13) and (15) the coefficients remain the same. It results from this fact that one can define in a general form the dependence of the coefficients appearing after the application of the GM decomposition at any arbitrary decomposition-level on some coefficients appearing in (5). Specifically, any arbitrary coefficient appearing in a decomposition-level may be defined as a function, having as index a sequence of natural numbers that correspond to the specific coefficients in (5), on which this coefficient depends. It follows from the structure of the Theorem 2 that the indices of all coefficients that appear at a particular inequality have the same length.

The index of every coefficient that appears at any decomposition-level has at least length 1, so it may be written as $m\bar{l}$, where $m \in \mathrm{N}$ and $\bar{l}$ is a sequence of length at least zero. Whenever this index has length at least two, it may be written as $m\bar{l}n$, where $m, n \in \mathrm{N}$. At any arbitrary decomposition-level, a coefficient, which has as index a sequence of $j \geq 2$ natural numbers, is denoted as $a_{m\,\bar{l}\,n}(t)$, where $m, n \in \mathrm{N}$ and $\bar{l}$ is a sequence of $j-2$ natural numbers. Similarly, $b_{\bar{l}\,n}(t)$ denotes the corresponding constant term in the same inequality, where a coefficient $a_{m\,\bar{l}\,n}(t)$ appears.

Below, in Definition 2, the coefficients $a_{m\,\bar{l}\,n}(t)$ and $b_{\bar{l}\,n}(t)$ are expressed recursively having as initial conditions $a_i(t)$ and $b(t)$. In Definition 2 the general case is presented, where the indices of the coefficients have at least length 2 and the index of the corresponding constant term has at least length 1, since the trivial case has already been presented in (5).

According to the GM decomposition, the sets $P_{m\,\bar{l}\,n}$ on which the corresponding inequality is defined, may be calculated recursively. Moreover, $R^1_{m\,\bar{l}\,n}, R^2_{m\,\bar{l}\,n}, R^3_{m\,\bar{l}\,n}$ denote the sets, on which a specific coefficient in the corresponding inequality is zero-, positive- or negative-valued. Finally, the auxiliary sets $S^1_{m\,\bar{l}\,n}, S^2_{m\,\bar{l}\,n}, S^3_{m\,\bar{l}\,n}$ may be defined as a generalization of S_i^1, S_i^2, S_i^3, denoting the sets, on which the corresponding coefficient $a_{m\,\bar{l}\,n}(t)$ is zero-, positive- and negative-valued. All these definitions are represented in the following Definition 2 and are of critical importance for the analysis concerning the reduction of a given arbitrary inequality of the form (5) to the implicit inequalities.

Definition 2. The sets $S^i_{m\,\bar{l}\,n}$, $i = 1, 2, 3$ are stepwise defined in terms of $a_{m\,\bar{l}\,n}(t)$ and $S^i_{n\,\bar{l}}$, $i = 2, 3$. Also, the coefficients $a_{m\,\bar{l}\,n}(t)$ and $b_{\bar{l}\,n}(t)$ are recursively defined in terms of $a_{m\,\bar{l}}(t)$, $a_{n\,\bar{l}}(t)$, $b_{\bar{l}}(t)$ and $S^i_{n\,\bar{l}}$, $i = 2, 3$, as follows:

Initial Conditions

$$S_j^i, \quad i = 1, 2, 3, \quad a_j(t), \quad b(t), \quad j = 1, 2, ..., k.$$

Recursions

$$a_{m\,\bar{l}\,n}\left(\bar{t}\right) := \frac{a_{m\,\bar{l}}(t_2)}{\left|a_{n\,\bar{l}}(t_2)\right|} + \frac{a_{m\,\bar{l}}(t_3)}{\left|a_{n\,\bar{l}}(t_3)\right|}, \quad b_{\bar{l}\,n}\left(\bar{t}\right) := \frac{b_{\bar{l}}(t_2)}{\left|a_{n\,\bar{l}}(t_2)\right|} + \frac{b_{\bar{l}}(t_3)}{\left|a_{n\,\bar{l}}(t_3)\right|}, \quad \forall\, \bar{t} = (t_2, t_3) \in S^2_{n\,\bar{l}} \times S^3_{n\,\bar{l}},$$

$$\begin{aligned} S^1_{m\,\bar{l}\,n} &:= \left\{ t \in S^2_{n\,\bar{l}} \times S^3_{n\,\bar{l}} : a_{m\,\bar{l}\,n}(t) = 0 \right\}, S^2_{m\,\bar{l}\,n} := \left\{ t \in S^2_{n\,\bar{l}} \times S^3_{n\,\bar{l}} : a_{m\,\bar{l}\,n}(t) > 0 \right\}, \\ S^3_{m\,\bar{l}\,n} &:= \left\{ t \in S^2_{n\,\bar{l}} \times S^3_{n\,\bar{l}} : a_{m\,\bar{l}\,n}(t) < 0 \right\}, \end{aligned}$$

where $m, n \in \mathrm{N}$, $\bar{l} = l_1 l_2 ... l_j \in \mathrm{N}^j$ for every $j \in \mathrm{N}_0 = \mathrm{N} \cup \{0\}$, with $l_1, l_2, ..., l_j \in \mathrm{N}$ pair wise distinct and $\bar{l} = \emptyset$ for $j = 0$.

Definition 3. The sets $P_{m\,\bar{l}\,n}$ and $R^1_{m\,\bar{l}\,n}, R^2_{m\,\bar{l}\,n}, R^3_{m\,\bar{l}\,n}$ are recursively defined as follows:

Initial Conditions

$$P_{1,2} = T, \text{ for } m = 1,\ n = m+1 = 2 \text{ and } \bar{l} = \emptyset,$$

$$R^i_{1,2} = S^i_1,\ i = 1, 2, 3.$$

Recursions

$$P_{m\,\bar{l}\,n} = \left\{ \begin{array}{ll} R^2_{(m-i)\left(\bar{l}\setminus(m-i)\right)m} \times R^3_{(m-i)\left(\bar{l}\setminus(m-i)\right)m} & , \text{ if } \bar{l} = \overline{l'}\,(m-i) \\ \bigcup\limits_{i=1}^{m-1} R^1_{(m-i)\,\bar{l}\,m} & , \text{ if } (m-i) \notin \bar{l} \end{array} \right\}, \text{ if } n = m+1, \text{ with } m > 1,$$

$$P_{m\,\bar{l}\,n} = \left\{ \begin{array}{ll} \left[P_{m\left(\bar{l}\setminus(n-1)\right)(n-1)} \cap S^2_{(n-1)\left(\bar{l}\setminus(n-1)\right)} \right] \times & \\ \times \left[P_{m\left(\bar{l}\setminus(n-1)\right)(n-1)} \cap S^3_{(n-1)\left(\bar{l}\setminus(n-1)\right)} \right] & , \text{ if } \bar{l} = \overline{l'}\,(n-1) \\ P_{m\,\bar{l}\,(n-1)} \cap S^1_{(n-1)\,\bar{l}} & , \text{ if } (n-1) \notin \bar{l} \end{array} \right\}, \text{ if } n > m+1,$$

$$R^1_{m\,\bar{l}\,n} = \left\{ t \in P_{m\,\bar{l}\,n} : a_{m\,\bar{l}}(t) = 0 \right\}, R^2_{m\,\bar{l}\,n} = \left\{ t \in P_{m\,\bar{l}\,n} : a_{m\,\bar{l}}(t) > 0 \right\},$$
$$R^3_{m\,\bar{l}\,n} = \left\{ t \in P_{m\,\bar{l}\,n} : a_{m\,\bar{l}}(t) < 0 \right\}.$$

In the sequel, we denote as "first index part" of a function the first integer that appears in its index, which is a sequence of natural numbers and as "second index part" the rest sequence of the index. Thus, the first index part of $a_{m\,\bar{l}}(t)$ is the integer m and the second index part is $\bar{l}$.

Lemma 2. *1. the coefficients and the corresponding constant terms have the form of $a_{m\,\bar{l}}(t)$ and $b_{\bar{l}}(t)$ respectively, as defined in Definition 2,*

2. *the indices of all coefficients coincide, except for their first part,*

3. *the indices of all coefficients have the same length,*

4. *the common second index part of them is exactly the index of the corresponding constant term,*

5. *whenever the variable $x_r,\ r \in \{1, 2, ..., k\}$ appears, the first index part m of the coefficient of x_r, remains constant and equal to $k+1-r$, i.e. in the inequality appears the product $a_{(k+1-r)\,\bar{l}\,n}(t)\, x_r$ and*

6. *all indices r of $x_r,\ r \in \{1, 2, ..., k\}$ that appear are either:*

   - *successive natural numbers, or*
   - *successive natural numbers except for the most right one, which can be arbitrary bigger than the others.*

Proof. The proof is presented in Appendix 1. □

Corollary 1. *An inequality at an arbitrary decomposition-level may be uniquely specified only in terms of the indices of the two most right coefficients that appear in the particular inequality.*

Proof. Suppose that the indices of the two most right coefficients that appear in a particular inequality are known, i.e. $a_{n\,\bar{l}}(t)$ and $a_{m\,\bar{l}}(t)$, with $n \geq m+1$. Then, due to Lemma 2, the index of the constant term and the second index part of all the coefficients is equal to $\bar{l}$. The first index part of every of the rest coefficients, i.e. the coefficient of x_r, is equal to $k+1-r$. Also, due to Lemma 2, all indices r of x_r, $r \in \{1,2,...,k\}$ that appear at the left of the first two known coefficients are successive natural numbers. Thus, the only inequality, that has $a_{n\,\bar{l}}(t)$ and $a_{m\,\bar{l}}(t)$ as the two most right coefficients is:

$$a_{k\,\bar{l}}(t)\,x_1 + a_{k-1,\,\bar{l}}(t)\,x_2 + \cdots + a_{n+1,\,\bar{l}}(t)\,x_{k-n} + a_{n\,\bar{l}}(t)\,x_{k+1-n} + a_{m\,\bar{l}}(t)\,x_{k+1-m} \geq b_{\bar{l}}(t)\,. \tag{19}$$

$\square$

Lemma 3. *The set $P_{m\,\bar{l}\,n}$ is the set on which inequality (19) is defined.*

Proof. The proof is presented in Appendix 2. $\square$

In the sequel, Theorem 3 is presented. The implicit inequalities in Theorem 3 provide analytically the ranges, where the variables x_r, $r=1,2,...,k$ lie, provided that the initial inequality (5) has at least one solution.

Theorem 3. *Applying successively $(k-1)$ times the GM decomposition and then one time the decomposition of Lemma 1 to the given inequality (5), we obtain the following set of inequalities, for every $r \in \{1,2,...,k\}$:*

$$\max_{\bar{l}}\left\{\sup_{t\in R^1_{(k-r+1)\,\bar{l}\,(k+1)}}\left\{b_{\bar{l}}(t)\right\}\right\} \leq 0, \tag{20}$$

$$\max_{\bar{l}}\left\{\sup_{t\in R^2_{(k-r+1)\,\bar{l}\,(k+1)}}\left\{\frac{b_{\bar{l}}(t)}{a_{(k-r+1)\,\bar{l}}(t)}\right\}\right\} \leq x_r \leq \min_{\bar{l}}\left\{\inf_{t\in R^3_{(k-r+1)\,\bar{l}\,(k+1)}}\left\{\frac{b_{\bar{l}}(t)}{a_{(k-r+1)\,\bar{l}}(t)}\right\}\right\}, \tag{21}$$

where maxima and minima are taken over every possible $\bar{l} = \overline{l^a}\,\overline{l^b} \in \mathrm{N}^{j_1+j_2}$, with $\overline{l^a} = l^a_1 l^a_2 ... l^a_{j_1} \in \mathrm{N}^{j_1}$, $\overline{l^b} = l^b_1 l^b_2 ... l^b_{j_2} \in \mathrm{N}^{j_2}$, $j_1 \in \{0,1,...,k-r\}$, $j_2 \in \{0,1,...,r-1\}$, such that:

$$\begin{aligned}
&l^a_1 \in \{1,...,k-r\}\,,\ l^b_1 \in \{k-r+2,...,k\}\,,\\
&l^a_j \in \left(\bigcap_{i=1}^{j-1}\{l^a_i-1,...,k-r\}\right) \setminus \{l^a_1,...,l^a_{j-1}\}\,, for\ 2 \leq j \leq j_1,\\
&l^b_j \in \{l^b_{j-1}+1,...,k\}\,, for\ 2 \leq j \leq j_2.
\end{aligned}$$

Proof. The given inequality (5) is decomposed initially into the four inequalities (13)-(16). Then the application of the GM decomposition to (13)-(16), produces a quadruplet of equivalent inequalities for each one of them and in total 4^2 inequalities. Proceeding in the same way and decomposing the 4^2

inequalities, we arrive at 4^3 inequalities and so on. In general, at the j^{th} decomposition-level 4^j inequalities are produced.

After $(k-1)$ successively applications of the GM decomposition to the initial inequality (5), as described above, we obtain an inequality with only one variable x_r; $r \in \{1, 2, ..., k\}$ and corresponding coefficient $a_{(k-r+1)\,\bar{l}}(t)$, for some appropriate $\bar{l}$, while a second one does not exist at all, since all the others have been eliminated during the successive applications of the GM decomposition. Considering in this inequality a zero-valued coefficient of an imaginary variable x_0 as the second one from the right, having $(k+1)$ as first index part, it is seen that the definition domain of this inequality is $P_{(k-r+1)\,\bar{l}\,(k+1)}$, for some appropriate $\bar{l}$. Indeed, $P_{m\,\bar{l}\,n}$ depends only on the coefficients having $i \in \{1, 2, \cdots, n-1\}$ and not n as first index part, as can be seen in Definition 3. Thus, after $(k-1)$ successively applications of the GM decomposition to the initial inequality (5) the following inequalities may be obtained:

$$a_{(k-r+1)\,\bar{l}}(t)\, x_r \geq b_{\bar{l}}(t)\,, \quad \forall t \in P_{(k-r+1)\,\bar{l}\,(k+1)}, \tag{22}$$

for every appropriate $\bar{l} = l_1 l_2 ... l_j \in \mathrm{N}^j, j \geq 0$.

In Appendix 3 it is proved that the possible integer-sequences $\bar{l}$ that may appear in $P_{(k-r+1)\,\bar{l}\,(k+1)}$ are exactly those of the form $\bar{l} = \overline{l^a}\,\overline{l^b} \in \mathrm{N}^{j_1+j_2}$, with $\overline{l^a}$ and $\overline{l^b}$ as given in the statement of Theorem 3.

Now, applying the special decomposition of Lemma 1 to (22), it results:

$$0 \geq b_{\bar{l}}(t)\,, \quad \forall t \in R^1_{(k-r+1)\,\bar{l}\,(k+1)}$$

$$\frac{b_{\bar{l}}(t_2)}{a_{(k-r+1)\,\bar{l}}(t_2)} \leq x_r \leq \frac{b_{\bar{l}}(t_3)}{a_{(k-r+1)\,\bar{l}}(t_3)}, \forall\,(t_2, t_3) \in R^2_{(k-r+1)\,\bar{l}\,(k+1)} \times R^3_{(k-r+1)\,\bar{l}\,(k+1)}$$

or equivalently:

$$\sup_{t \in R^1_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ b_{\bar{l}}(t) \right\} \leq 0$$

$$\sup_{t \in R^2_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{\bar{l}}(t)}{a_{(k-r+1)\,\bar{l}}(t)} \right\} \leq x_r \leq \inf_{t \in R^3_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{\bar{l}}(t)}{a_{(k-r+1)\,\bar{l}}(t)} \right\}$$

for every appropriate $\bar{l}$, as described above, or equivalently we obtain (20) and (21). Thus, Theorem 3 is proved. □

4 Main Results

In the following, in Theorem 4, the results obtained in Theorem 3 are used for deriving the necessary and sufficient conditions for the existence of a solution of a system of LMIs in Toeplitz form, along with some bounds of the solution, if such exists.

Theorem 4. *The necessary and sufficient conditions for the existence of a solution* $\mathbf{x} = [x_1, x_2, ..., x_N]^T \in \mathrm{R}^N$*, satisfying the inequality:*

$$\mathbf{A}(t)\,\mathbf{x} = \begin{bmatrix} a_1(t) & 0 & \cdots & 0 \\ a_2(t) & a_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_N(t) & a_{N-1}(t) & \cdots & a_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \geq \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_N(t) \end{bmatrix} = \mathbf{b}(t)\,, \forall t \in T, \tag{23}$$

are the following:

$$\max_{\substack{k\in\{r,r+1,\ldots,N\}\\ \bar{l}}} \left\{ \sup_{\bar{t}\in R^1_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ b_{k\,\bar{l}}\left(\bar{t}\right) \right\} \right\} \leq 0, \forall r \in \{1, 2, ..., N\}\,, \tag{24}$$

$$\max_{\substack{k\in\{r,r+1,\ldots,N\}\\ \bar{l}}} \left\{ \sup_{\bar{t}\in R^2_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\left(\bar{t}\right)}{a_{(k-r+1)\,\bar{l}}\left(\bar{t}\right)} \right\} \right\} \leq$$
$$\leq \min_{\substack{k\in\{r,r+1,\ldots,N\}\\ \bar{l}}} \left\{ \inf_{\bar{t}\in R^3_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\left(\bar{t}\right)}{a_{(k-r+1)\,\bar{l}}\left(\bar{t}\right)} \right\} \right\}, \forall r \in \{1, 2, ..., N\} \tag{25}$$

and the solution $\mathbf{x} = [x_1, x_2, ..., x_r, ..., x_N]^T \in \mathrm{R}^N$ *is bounded by:*

$$x_r \in \left[\max_{\substack{k\in\{r,r+1,\ldots,N\}\\ \bar{l}}} \left\{ \sup_{\bar{t}\in R^2_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\left(\bar{t}\right)}{a_{(k-r+1)\,\bar{l}}\left(\bar{t}\right)} \right\} \right\}, \right.$$
$$\left. \min_{\substack{k\in\{r,r+1,\ldots,N\}\\ \bar{l}}} \left\{ \inf_{\bar{t}\in R^3_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\left(\bar{t}\right)}{a_{(k-r+1)\,\bar{l}}\left(\bar{t}\right)} \right\} \right\} \right] \tag{26}$$

where maxima and minima are taken over k *and over every possible* $\bar{l} = \overline{l^a}\,\overline{l^b} \in \mathrm{N}^{j_1+j_2}$*, with* $\overline{l^a} = l^a_1 l^a_2 ... l^a_{j_1} \in \mathrm{N}^{j_1}$*,* $\overline{l^b} = l^b_1 l^b_2 ... l^b_{j_2} \in \mathrm{N}^{j_2}$*,* $j_1 \in \{0, 1, ..., k-r\}$*,* $j_2 \in \{0, 1, ..., r-1\}$*, such that:*

$$\begin{aligned} & l^a_1 \in \{1, ..., k-r\}\,, \quad l^b_1 \in \{k-r+2, ..., k\}\,, \\ & l^a_j \in \left(\bigcap_{i=1}^{j-1} \{l^a_i - 1, ..., k-r\} \right) \setminus \left\{ l^a_1, ..., l^a_{j-1} \right\}, for\ 2 \leq j \leq j_1, \\ & l^b_j \in \left\{ l^b_{j-1} + 1, ..., k \right\}, for\ 2 \leq j \leq j_2. \end{aligned}$$

Proof. The LMIs in (23) are written for $k = 1, 2, ..., N$ and $\forall t \in T$ in the form:

$$\sum_{i=1}^{k} a_{k-i+1}(t)\, x_i \geq b_k(t)\,;\ \ k = 1, 2, ..., N \tag{27}$$

It is seen from (27), that for any $r \in \{1, 2, ..., N\}$, the restrictions on x_r are imposed only from the inequalities of the rows $r, r+1, ..., N$. For the k^{th}, $k = r, r+1, ..., N$ inequality, the restrictions on x_r are described in (20) and (21). Summarizing the restrictions on x_r from the inequalities of the rows $r, r+1, ..., N$ of (27) and considering all $r \in \{1, 2, ..., N\}$, it results that the necessary and sufficient conditions, such that a solution $\mathbf{x} = [x_1, x_2, ..., x_r, ..., x_N]^T \in \mathrm{R}^N$ exists, satisfying (27) are the following:

$$\max_{\substack{k\in\{r,r+1,\dots,N\}\\ \bar{l}}} \left\{ \sup_{\bar{t}\in R^1_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ b_{k\,\bar{l}}\,(\bar{t}) \right\} \right\} \leq 0,\ \forall r \in \{1, 2, ..., N\}, \tag{28}$$

$$\max_{\substack{k\in\{r,r+1,\dots,N\}\\ \bar{l}}} \left\{ \sup_{t_2\in R^2_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\,(t_2)}{a_{(k-r+1)\,\bar{l}}\,(t_2)} \right\} \right\} \leq x_r \leq$$

$$\leq \min_{\substack{k\in\{r,r+1,\dots,N\}\\ \bar{l}}} \left\{ \inf_{t_3\in R^3_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\,(t_3)}{a_{(k-r+1)\,\bar{l}}\,(t_3)} \right\} \right\}, \forall r \in \{1, 2, ..., N\}, \tag{29}$$

where maxima and minima are taken over k and over every appropriate $\bar{l}$, as described above. In (29) a $x_r \in \mathrm{R}$ exists if and only if the upper bound of x_r is greater than or equal to the corresponding lower bound. Therefore, the necessary and sufficient conditions, such that some $\mathbf{x} \in \mathrm{R}^N$ exists, satisfying (27), are the inequalities (24) and (25).

Now, suppose that conditions (24) and (25) are satisfied, so that a solution $\mathbf{x} = [x_1, x_2, ..., x_r, ..., x_N]^T \in \mathrm{R}^N$ of the system in (27) exists. We will find out where all these solutions lie. The conditions (28) and (29) have been derived by using only the decompositions of Theorem 2 and Lemma 1. In Theorem 2 (Lemma 1) it is proved that the solution of an inequality is also a solution of the four (two) produced inequalities. Continuing in this way, it results that $\mathbf{x} \in \mathrm{R}^N$ satisfies also the produced set of inequalities in (29). Therefore, it results from (29) that the arbitrary component x_r of the solution $\mathbf{x} \in \mathrm{R}^N$, if such exists, lies in the following set:

$$x_r \in \left[\max_{\substack{k\in\{r,r+1,\dots,N\}\\ \bar{l}}} \left\{ \sup_{t_2\in R^2_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\,(t_2)}{a_{(k-r+1)\,\bar{l}}\,(t_2)} \right\} \right\}, \right.$$
$$\left. \min_{\substack{k\in\{r,r+1,\dots,N\}\\ \bar{l}}} \left\{ \inf_{t_3\in R^3_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\,(t_3)}{a_{(k-r+1)\,\bar{l}}\,(t_3)} \right\} \right\} \right]$$

where maxima and minima are taken over k and over every possible $\bar{l} = \overline{l^a}\,\overline{l^b} \in \mathrm{N}^{j_1+j_2}$, with $\overline{l^a}$ and $\overline{l^b}$ as given in the statement of Theorem 4. Thus Theorem 4 is proved. □

Throughout the paper the general case has been considered, where all the sets $S^1_{m\,\bar{l}\,n}$, $S^2_{m\,\bar{l}\,n}$, $S^3_{m\,\bar{l}\,n}$, $R^1_{m\,\bar{l}\,n}$, $R^2_{m\,\bar{l}\,n}$, $R^3_{m\,\bar{l}\,n}$ (and thus also $P_{m\,\bar{l}\,n}$) that occur are not empty. If some of these are empty, then (24) and (25) degenerate, thus reducing the restrictions of the desired solutions. If one side of (25)

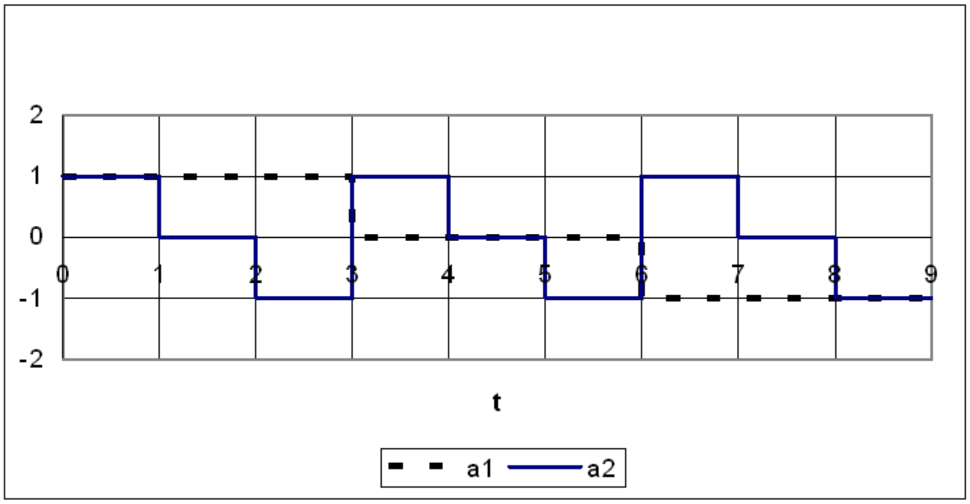

Figure 1: The functions $a_1(t)$ and $a_2(t)$.

vanishes for some $r \in \{1, 2, ..., N\}$, then it should be required that the other side is finite, since in the opposite case no finite real value for x_r exists. Assuming that all elements of $\mathbf{A}$ and $\mathbf{b}$ are bounded in T, then both sides in (25) are finite for every $r \in \{1, 2, ..., N\}$; thus an inequality can be ignored, whenever one side of it vanishes. In this case the necessary and sufficient conditions for the existence of the solution of the system, as well as the restrictions of the final solution, are also derived without any other modification.

Example

Consider the linear system for $N = 2$:

$$\begin{bmatrix} a_1(t) & 0 \\ a_2(t) & a_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1(t)\, x_1 \\ a_2(t)\, x_1 + a_1(t)\, x_2 \end{bmatrix} \geq \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}, \forall t \in T = [0, 9),$$

where:

$$a_1(t) = \begin{cases} 1, & t \in [0,3) \\ 0, & t \in [3,6) \\ -1, & t \in [6,9) \end{cases}, \; a_2(t) = \begin{cases} 1, & t \in [0,1) \cup [3,4) \cup [6,7) \\ 0, & t \in [1,2) \cup [4,5) \cup [7,8) \\ -1, & t \in [2,3) \cup [5,6) \cup [8,9) \end{cases}, \; b_1(t) = t^2 - 82, \; b_2(t) = t - 10.$$

The functions $a_1(t)$ and $a_2(t)$, for all $t \in T$, are graphically shown in Figure 1. Thus, according to Definition 1, it holds:

$$S_1^1 = [3, 6), \quad S_1^2 = [0, 3), \quad S_1^3 = [6, 9),$$

$$S_2^1 = [1, 2) \cup [4, 5) \cup [7, 8), \quad S_2^2 = [0, 1) \cup [3, 4) \cup [6, 7), \quad S_2^3 = [2, 3) \cup [5, 6) \cup [8, 9).$$

It is seen from (24) and (25) that for every $r \in \{1,2\}$ only $k \in \{r, r+1, \cdots, N\}$ is considered. For $r=1$ we obtain $k \in \{1,2\}$. For $k=1$ the only possible $\bar{l}$ is $\emptyset$. For $k=2$ the possible $\bar{l}$'s are $\emptyset$ and 1 (as described in Theorem 4). For $r=2$ we obtain only $k=2$. Now, the possible $\bar{l}$'s are $\emptyset$ and 2. The conditions (24) and (25) for $r=1$ and $r=2$ take the form of (30),(31) and (32),(33) respectively, as follows:

$$\max\left\{\sup_{\bar{t}\in R^1_{1,2}}\left\{b_1\left(\bar{t}\right)\right\},\ \sup_{\bar{t}\in R^1_{2,3}}\left\{b_2\left(\bar{t}\right)\right\},\ \sup_{\bar{t}\in R^1_{2,1,3}}\left\{b_{2,1}\left(\bar{t}\right)\right\}\right\}\le 0, \tag{30}$$

$$\max\left\{\sup_{\bar{t}\in R^1_{1,3}}\left\{b_2\left(\bar{t}\right)\right\},\ \sup_{\bar{t}\in R^1_{1,2,3}}\left\{b_{2,2}\left(\bar{t}\right)\right\}\right\}\le 0, \tag{31}$$

$$\begin{aligned}\max&\left\{\sup_{\bar{t}\in R^2_{1,2}}\left\{\frac{b_1\left(\bar{t}\right)}{a_1\left(\bar{t}\right)}\right\},\ \sup_{\bar{t}\in R^2_{2,3}}\left\{\frac{b_2\left(\bar{t}\right)}{a_2\left(\bar{t}\right)}\right\},\ \sup_{\bar{t}\in R^2_{2,1,3}}\left\{\frac{b_{2,1}\left(\bar{t}\right)}{a_{2,1}\left(\bar{t}\right)}\right\}\right\}\le\\ &\le\min\left\{\inf_{\bar{t}\in R^3_{1,2}}\left\{\frac{b_1\left(\bar{t}\right)}{a_1\left(\bar{t}\right)}\right\},\ \inf_{\bar{t}\in R^3_{2,3}}\left\{\frac{b_2\left(\bar{t}\right)}{a_2\left(\bar{t}\right)}\right\},\ \inf_{\bar{t}\in R^3_{2,1,3}}\left\{\frac{b_{2,1}\left(\bar{t}\right)}{a_{2,1}\left(\bar{t}\right)}\right\}\right\},\end{aligned} \tag{32}$$

$$\max\left\{\sup_{\bar{t}\in R^2_{1,3}}\left\{\frac{b_2\left(\bar{t}\right)}{a_1\left(\bar{t}\right)}\right\},\ \sup_{\bar{t}\in R^2_{1,2,3}}\left\{\frac{b_{2,2}\left(\bar{t}\right)}{a_{1,2}\left(\bar{t}\right)}\right\}\right\}\le\min\left\{\inf_{\bar{t}\in R^3_{1,3}}\left\{\frac{b_2\left(\bar{t}\right)}{a_1\left(\bar{t}\right)}\right\},\ \inf_{\bar{t}\in R^3_{1,2,3}}\left\{\frac{b_{2,2}\left(\bar{t}\right)}{a_{1,2}\left(\bar{t}\right)}\right\}\right\}. \tag{33}$$

In order to compute (30)-(33), it is required to compute first the following sets and functions:

$$P_{1,2}=T=[0,9)\,,\quad P_{2,3}=R^1_{1,2}=S^1_1=[3,6)\,,\quad P_{2,1,3}=R^2_{1,2}\times R^3_{1,2}=S^2_1\times S^3_1=[0,3)\times[6,9)\,,$$

$$P_{1,3}=P_{1,2}\cap S^1_2=S^1_2=[1,2)\times[4,5)\times[7,8)\,,$$

$$P_{1,2,3}=\left[P_{1,2}\cap S^2_2\right]\times\left[P_{1,2}\cap S^3_2\right]=S^2_2\times S^3_2=\left([0,1)\cup[3,4)\cup[6,7)\right)\times\left([2,3)\cup[5,6)\cup[8,9)\right),$$

$$b_{2,1}\left(t_2,t_3\right)=\frac{b_2\left(t_2\right)}{\left|a_1\left(t_2\right)\right|}+\frac{b_2\left(t_3\right)}{\left|a_1\left(t_3\right)\right|}=\frac{t_2-10}{|1|}+\frac{t_3-10}{|-1|}=t_2+t_3-20,\forall\left(t_2,t_3\right)\in S^2_1\times S^3_1,$$

$$b_{2,2}\left(t_2,t_3\right)=\frac{b_2\left(t_2\right)}{\left|a_2\left(t_2\right)\right|}+\frac{b_2\left(t_3\right)}{\left|a_2\left(t_3\right)\right|}=\frac{t_2-10}{|1|}+\frac{t_3-10}{|-1|}=t_2+t_3-20,\forall\left(t_2,t_3\right)\in S^2_2\times S^3_2,$$

$$a_{2,1}\left(t_2,t_3\right)=\frac{a_2\left(t_2\right)}{\left|a_1\left(t_2\right)\right|}+\frac{a_2\left(t_3\right)}{\left|a_1\left(t_3\right)\right|}=\frac{a_2\left(t_2\right)}{|1|}+\frac{a_2\left(t_3\right)}{|-1|}=a_2\left(t_2\right)+a_2\left(t_3\right),\forall\left(t_2,t_3\right)\in S^2_1\times S^3_1,$$

$$a_{1,2}\left(t_2,t_3\right)=\frac{a_1\left(t_2\right)}{\left|a_2\left(t_2\right)\right|}+\frac{a_1\left(t_3\right)}{\left|a_2\left(t_3\right)\right|}=\frac{a_1\left(t_2\right)}{|1|}+\frac{a_1\left(t_3\right)}{|-1|}=a_1\left(t_2\right)+a_1\left(t_3\right),\forall\left(t_2,t_3\right)\in S^2_2\times S^3_2,$$

$$R^1_{1,2}=\left\{t\in P_{1,2}:a_1\left(t\right)=0\right\}=\left\{t\in T:a_1\left(t\right)=0\right\}=S^1_1=[3,6)\,,R^2_{1,2}=S^2_1=[0,3)\,,$$
$$R^3_{1,2}=S^3_1=[6,9)\,,$$

$$R_{2,3}^{1}=\{t\in P_{2,3}:a_{2}(t)=0\}=\{t\in[3,6):a_{2}(t)=0\}=[4,5)\,,\quad R_{2,3}^{2}=[3,4)\,,\quad R_{2,3}^{3}=[5,6)\,,$$

$$\begin{aligned}R_{2,1,3}^{1}&=\{t\in P_{2,1,3}:a_{2,1}(t)=0\}=\{(t_{2},t_{3})\in[0,3)\times[6,9):a_{2}(t_{2})+a_{2}(t_{3})=0\}\\&=([0,1)\times[8,9))\cup([1,2)\times[7,8))\cup([2,3)\times[6,7))\,,\end{aligned}$$

$$R_{2,1,3}^{2}=([0,1)\times[6,7))\cup([0,1)\times[7,8))\cup([1,2)\times[6,7))\,,$$

$$R_{2,1,3}^{3}=([1,2)\times[8,9))\cup([2,3)\times[7,8))\cup([2,3)\times[8,9))\,,$$

$$R_{1,3}^{1}=\{t\in P_{1,3}:a_{1}(t)=0\}=\left\{t\in S_{2}^{1}:a_{1}(t)=0\right\}=[4,5)\,,$$

$$R_{1,3}^{2}=[1,2)\,,\quad R_{1,3}^{3}=[7,8)\,,$$

$$\begin{aligned}R_{1,2,3}^{1}&=\{t\in P_{1,2,3}:a_{1,2}(t)=0\}=\\&=\{(t_{2},t_{3})\in([0,1)\cup[3,4)\cup[6,7))\times([2,3)\cup[5,6)\cup[8,9)):a_{1}(t_{2})+a_{1}(t_{3})=0\}=\\&=([0,1)\times[8,9))\cup([3,4)\times[5,6))\cup([6,7)\times[2,3))\,,\end{aligned}$$

$$R_{1,2,3}^{2}=([0,1)\times[2,3))\cup([0,1)\times[5,6))\cup([3,4)\times[2,3))\,,$$

$$R_{1,2,3}^{3}=([3,4)\times[8,9))\cup([6,7)\times[5,6))\cup([6,7)\times[8,9))\,.$$

In the sequel, using the above quantities, we check whether inequalities (30)-(33) are satisfied. Indeed, (30)-(33) hold, since $\max\{-46,-5,-10\}=-5\leq 0$, $\max\{-5,-10\}=-5\leq 0$, $\max\{-73,-6,-11\}=-6\leq\min\{1,4,4\}=1$ and $\max\{-8,-8\}=-8\leq\min\{2,2\}=2$. Therefore a solution $x=\begin{bmatrix}x_{1}&x_{2}\end{bmatrix}^{T}\in\mathrm{R}^{2}$ exists, which is bounded by $x_{1}\in[-6,1]$, $x_{2}\in[-8,2]$, as it follows from (26).

The exact set of solutions of the system and the bounds of these solutions, as given above, are graphically shown in Figure 2. The rectangle produced from these bounds is the smallest possible, since its erosion leads to loss of solutions.

Appendix 1: Proof of Lemma 2

The proof is done by induction. In (5) all coefficients are of the form $a_{i}(t)\,,i\in\{1,2,...,k\}$ and the constant term is $b(t)$. Here $\overline{l}=\emptyset$ is the index of the constant term and also the second index part of every coefficient. All the indices of the coefficients coincide, except of their first index part i of them and thus they have all the same length. Also, the index of the coefficient of the variable $x_{r},\ r\in\{1,2,...,k\}$ is equal to $k+1-r$ and the indices r of $x_{r},\ r\in\{1,2,...,k\}$ are successive natural numbers. This shows the initialization of the induction procedure.

Suppose now that Lemma 2 holds until an arbitrary decomposition-level. Let we have at that level

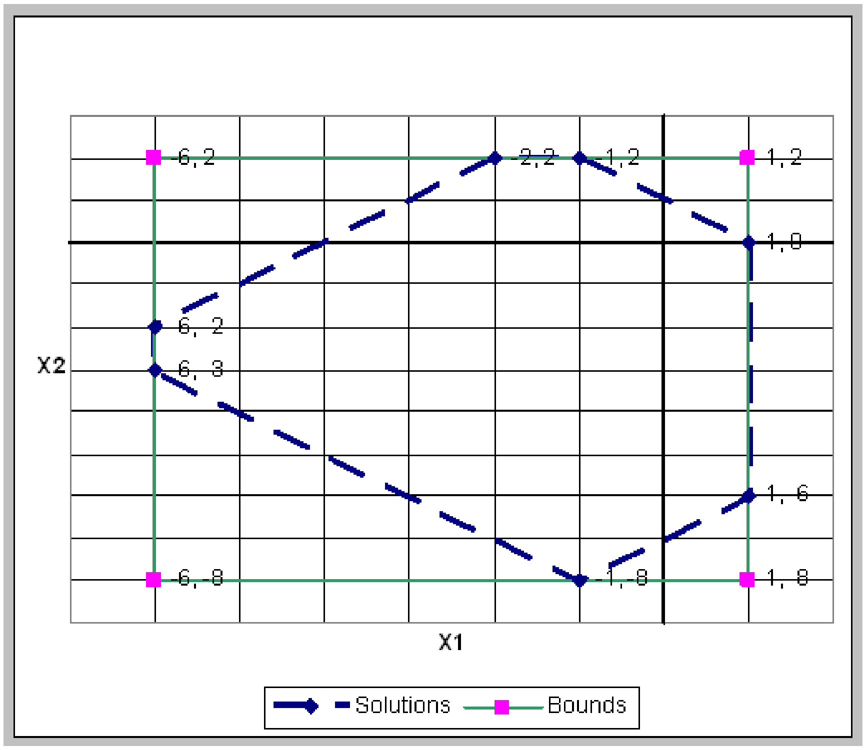

Figure 2: The set of the solutions of the system and their bounds.

the inequality:

$$a_{k\bar{l}}(t)\,x_1 + a_{k-1,\bar{l}}(t)\,x_2 + \cdots + a_{n+1,\bar{l}}(t)\,x_{k-n} + a_{n\bar{l}}(t)\,x_{k+1-n} + a_{m\bar{l}}(t)\,x_{k+1-m} \geq b_{\bar{l}}(t)\,, \forall t \in \Pi, \tag{A1.1}$$

for some set Π, where $n \geq m+1$. We apply now to (A1.1) the GM decomposition and we obtain from (13)-(16) respectively:

$$a_{k\bar{l}}(t)\,x_1 + \cdots + a_{n+1,\bar{l}}(t)\,x_{k-n} + a_{n\bar{l}}(t)\,x_{k+1-n} \geq b_{\bar{l}}(t)\,, \forall t \in \Pi_m^1 \tag{A1.2}$$

$$\begin{aligned}&\left[\frac{a_{k\bar{l}}(t_2)}{\left|a_{m\bar{l}}(t_2)\right|} + \frac{a_{k\bar{l}}(t_3)}{\left|a_{m\bar{l}}(t_3)\right|}\right] x_1 + \cdots + \left[\frac{a_{n+1,\bar{l}}(t_2)}{\left|a_{m\bar{l}}(t_2)\right|} + \frac{a_{n+1,\bar{l}}(t_3)}{\left|a_{m\bar{l}}(t_3)\right|}\right] x_{k-n} + \\ &+ \left[\frac{a_{n\bar{l}}(t_2)}{\left|a_{m\bar{l}}(t_2)\right|} + \frac{a_{n\bar{l}}(t_3)}{\left|a_{m\bar{l}}(t_3)\right|}\right] x_{k+1-n} \geq \left[\frac{b_{\bar{l}}(t_2)}{\left|a_{m\bar{l}}(t_2)\right|} + \frac{b_{\bar{l}}(t_3)}{\left|a_{m\bar{l}}(t_3)\right|}\right], \forall\,(t_2, t_3) \in \Pi_m^2 \times \Pi_m^3 \end{aligned} \tag{A1.3}$$

$$a_{k\bar{l}}(t)\,x_1 + \cdots + a_{n+1,\bar{l}}(t)\,x_{k-n} + a_{m\bar{l}}(t)\,x_{k+1-m} \geq b_{\bar{l}}(t)\,, \forall t \in \Pi_n^1 \tag{A1.4}$$

$$\left[\frac{a_{k\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|}+\frac{a_{k\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|}\right]x_1+\cdots+\left[\frac{a_{n+1,\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|}+\frac{a_{n+1,\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|}\right]x_{k-n}$$

$$+\left[\frac{a_{m\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|}+\frac{a_{m\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|}\right]x_{k+1-m}\geq\left[\frac{b_{\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|}+\frac{b_{\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|}\right],\forall(t_2,t_3)\in\Pi_n^2\times\Pi_n^3 \quad (A1.5)$$

where $\Pi_r^1=\left\{t\in\Pi:a_{r\bar{l}}(t)=0\right\}$, $\Pi_r^2=\left\{t\in\Pi:a_{r\bar{l}}(t)>0\right\}$, $\Pi_r^3=\left\{t\in\Pi:a_{r\bar{l}}(t)<0\right\}$ and $r\in\{m,n\}$.

The coefficients and the constant terms of the inequalities (A1.2)-(A1.5) coincide with the corresponding coefficients, as defined in of Definition 2. For example, in the second inequality above we have:

$$a_{k\bar{l}n}(\bar{t})=\frac{a_{k\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|}+\frac{a_{k\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|}\text{ and }b_{\bar{l}n}(\bar{t})=\frac{b_{\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|}+\frac{b_{\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|},\text{ where }\bar{t}=(t_2,t_3).$$

It is also clear that the indices of all the coefficients in a particular inequality of (A1.2)-(A1.5) coincide except for their first index part, they all have the same length and their common second part is exactly the index of the corresponding constant term. Also the first index part of the coefficient of the variable x_r in each one of the above four inequalities is equal to the first index part of the corresponding coefficient in (A1.1) and thus equal to the corresponding coefficient in (5).

Also, it is easily seen in Definition 2 that the sets $S^1_{m\bar{l}}, S^2_{m\bar{l}}, S^3_{m\bar{l}}$ are defined recursively and constitute a natural generalization of S_i^1, S_i^2, S_i^3. Thus, in the above inequalities it holds: $\Pi_m^1\subseteq S^1_{m\bar{l}}$, $\Pi_m^2\subseteq S^2_{m\bar{l}}$, $\Pi_m^3\subseteq S^3_{m\bar{l}}$, $\Pi_n^1\subseteq S^1_{n\bar{l}}$, $\Pi_n^2\subseteq S^2_{n\bar{l}}$ and $\Pi_n^3\subseteq S^3_{n\bar{l}}$, which means that the corresponding functions are properly defined.

It results from the above procedure that the length of the index of a coefficient increases if and only if one of the inequalities (A1.3) or (A1.5) appear; otherwise it remains constant. In addition, each time that some index increases, the increment equals the first index part of the coefficient of the variable, which vanishes in the inequality that appears.

In the whole procedure above, either the first most right coefficient, or the second one from the right, disappears, due to the inequalities (A1.2)-(A1.5). Continuing in this way, it results that all the indices r of x_r, $r\in\{1,2,...,k\}$ that appear in a particular iteration are either:

1. successive natural numbers, or

2. successive natural numbers except of the most right one, which can is arbitrary bigger than the others.

Appendix 2: Proof of Lemma 3

In the sequel, we call "parent" the inequality, from which the "present" inequality (19) is derived, after one application of the GM decomposition.

Initial Condition. In the given initial inequality (5), the two most right coefficients are $a_2(t)$ and $a_1(t)$. Thus this inequality is defined on the set $P_{1,2}$. The definition domain for this inequality is the whole T and thus the initial condition is $P_{1,2} = T$. Here $m = 1$, $l = \emptyset$, $n = m + 1 = 2$.

Case 1: $n = m + 1$. The case, where $m = 1$ is the initial condition. Suppose now that $m > 1$. This means that the two most right coefficients are $a_{(m+1)\bar{l}}$ and $a_{m\bar{l}}$. The present inequality (19) is defined on the set $P_{m\bar{l}(m+1)}$. The fact that the two most right coefficients have consecutive $m, n = m + 1$ first index parts, dictates that this inequality can be produced only by (13) or (14).

In the parent inequality, the first index part of the most right coefficient may take the values in $\{1, 2, ..., m - 1\}$, while the first index part of the second coefficient is m. Let $(m - i)$ be the first index part of the first coefficient in the parent inequality, for some $i \in \{1, 2, ..., m - 1\}$.

Now the following cases are discriminated:

1. Let $(m - i) \in \bar{l}$. The present inequality is produced from (14) and $(m - i)$ is the last integer that occurs in the sequence $\bar{l}$, i.e. $\bar{l} = \bar{l'}(m - i)$ for another sequence $\bar{l'}$. So, in the parent inequality, the common index part is $\bar{l}\backslash(m - i)$, which means that the two most right coefficients are $a_{m\left(\bar{l}\backslash(m-i)\right)}$ and $a_{(m-i)\left(\bar{l}\backslash(m-i)\right)}$. Thus, the parent inequality is defined on the set $P_{(m-i)\left(\bar{l}\backslash(m-i)\right)m}$ and it holds:
$$P_{m\bar{l}n} = R^2_{(m-i)\left(\bar{l}\backslash(m-i)\right)m} \times R^3_{(m-i)\left(\bar{l}\backslash(m-i)\right)m}.$$

2. Let $(m - i) \notin \bar{l}$. The present inequality is produced from (13), which means that in the parent inequality the two most right coefficients are $a_{m\bar{l}}$ and $a_{(m-i)\bar{l}}$. Thus, the parent inequality is defined on the set $P_{(m-i)\bar{l}m}$, which is possible for every $i \in \{1, 2, ..., m - 1\}$ and it holds:
$$P_{m\bar{l}n} = \bigcup_{i=1}^{m-1} R^1_{(m-i)\bar{l}m}.$$

Case 2: $n > m + 1$ In this case the two most right coefficients are $a_{n\bar{l}}$ and $a_{m\bar{l}}$, with $n > m + 1$. The fact that $n > m+1$ dictates that this inequality can be produced from the parent inequality only by (15) or (16).

Now the following cases are discriminated:

1. Let $(n - 1) \in \bar{l}$. The present inequality is produced from (16) and $(n - 1)$ is the last integer that occurs in the sequence $\bar{l}$, i.e. $\bar{l} = \bar{l'}(n - 1)$ for another sequence $\bar{l'}$. So, in the parent inequality, the common index part is $\bar{l}\backslash(n - 1)$, which means that the two most right coefficients are $a_{(n-1)\left(\bar{l}\backslash(n-1)\right)}$ and $a_{m\left(\bar{l}\backslash(n-1)\right)}$. Thus, the parent inequality is defined on the set $P_{m\left(\bar{l}\backslash(n-1)\right)(n-1)}$ and it holds:
$$P_{m\bar{l}n} = \left[P_{m\left(\bar{l}\backslash(n-1)\right)(n-1)} \cap S^2_{(n-1)\left(\bar{l}\backslash(n-1)\right)}\right] \times \left[P_{m\left(\bar{l}\backslash(n-1)\right)(n-1)} \cap S^3_{(n-1)\left(\bar{l}\backslash(n-1)\right)}\right].$$

2. Let $(n-1) \notin \overline{l}$. The present inequality is produced from (15), which means that in the parent inequality the two most right coefficients are $a_{(n-1)\,\overline{l}}$ and $a_{m\,\overline{l}}$. Thus, the parent inequality is defined on the set $P_{m\,\overline{l}\,(n-1)}$ and it holds:

$$P_{m\,\overline{l}\,n} = P_{m\,\overline{l}\,(n-1)} \cap S^1_{(n-1)\,\overline{l}}.$$

Appendix 3: Computation of all possible integer-sequences $\overline{l}$.

At first note that a value l_i in $\overline{l}$ denotes that somewhere during the iterations of the GM decomposition the coefficient with l_i as first index part i.e. the coefficient of x_{k-l_i+1}, has been eliminated from some of the inequalities (A1.3) or (A1.5).

There are two "blocks" of l_i in $\overline{l}$; those, which are smaller than $k-r+1$ and those, which are greater than $k-r+1$, since the value $l_i = k-r+1$ corresponds to an elimination of the coefficient of x_r that did not happen. The block with those l_i's that are smaller than $k-r+1$ appears first in $\overline{l}$ and the other block appears afterwards. Indeed, suppose there are some $l_j > k-r+1 > l_i$ for some $j < i$. This means that the coefficient of x_{k-l_j+1}, with l_j as first index part, has been eliminated in an inequality of the form (A1.3) or (A1.5), while the coefficient with l_i as first index part has not been eliminated yet (since $i > j$). Moreover, when the coefficient having $l_j > k-r+1$ as first index part is eliminated, then at least two coefficients on its right side appeared, each one having smaller first index part (that with l_i and that with $k-r+1$ as first index part respectively). However, this can never happen, due to the structure of the GM decomposition (always either the most right coefficient, or the second one is eliminated). Thus, the first "block" of l_i's comes first in the representation of $\overline{l}$.

Also, obviously, $l_i \neq l_j$ for l_i and l_j in $\overline{l}$. Thus, the length of the first and the second "blocks" of $\overline{l}$ are maximal $k-r$ and $r-1$ respectively and we may write $\overline{l} = \overline{l^a}\,\overline{l^b}$, where:

- $\overline{l^a} = l^a_1 l^a_2 ... l^a_{j_1}$, with: $l^a_1, l^a_2, ..., l^a_{j_1} \in \{1, 2, ..., k-r\}$, $j_1 \in \{0, 1, ..., k-r\}$,
- $\overline{l^b} = l^b_1 l^b_2 ... l^b_{j_2}$, with: $l^b_1, l^b_2, ..., l^b_{j_1} \in \{k-r+2, ..., k\}$, $j_2 \in \{0, 1, ..., r-1\}$.

In the sequel the possible values of $l^a_i,\ 1 \leq i \leq j_1$ and $l^b_i,\ 1 \leq i \leq j_2$ will be determined.

At first, consider $\overline{l^a} = l^a_1 l^a_2 ... l^a_{j_1}$. As the GM decomposition evolves, the inequalities (A1.2) or (A1.4) can appear many times, until either (A1.3) or (A1.5) occur. When one of the inequalities (A1.3) or (A1.5) occurs for first time, the coefficient having l^a_1 as first index part is eliminated. Then, the next time that one of the inequalities (A1.3) or (A1.5) occurs, the coefficient having l^a_2 as first index part is eliminated. There are two possibilities for l^a_1; the coefficient with l^a_1 as first index part is either the first, or the second one from the right in the inequality, at which one of the inequalities (A1.3) or (A1.5) occurs for first time. If it is the first one (so there are no other coefficients at its right side), then l^a_2 can be at least equal to $l^a_1 + 1$. If it is the second one (so there is exactly one other coefficient at its right side), then l^a_2 can be

at least $l_1^a - 1$ (i.e. l_1^a has been produced from the inequality (A1.5) and l_2^a from the inequality (A1.3)). Summarizing, l_2^a can take any value of the set $\{l_1^a - 1, ..., k-r\} \setminus \{0, l_1^a\}$, since l_1^a and l_2^a are distinct and different from zero. Continuing in a similar way, it results:

$$l_j^a \in \left(\left\{l_{j-1}^a - 1, ..., k-r\right\} \cap \left\{l_{j-2}^a - 1, ..., k-r\right\} \cap ... \cap \left\{l_1^a - 1, ..., k-r\right\}\right) \setminus \left\{0, l_1^a, ..., l_{j-1}^a\right\},$$

or equivalently:

$$l_1^a \in \{1, ..., k-r\} \text{ and } l_j^a \in \left(\bigcap_{i=1}^{j-1} \{l_i^a - 1, ..., k-r\}\right) \setminus \left\{0, l_1^a, ..., l_{j-1}^a\right\}, 2 \le j \le j_1.$$

Now, consider $\overline{l^b} = l_1^b l_2^b ... l_{j_2}^b$. After eliminating all coefficients at the right side of the coefficient of x_r, no elimination from (A1.2) or (A1.3) is possible, since the most right coefficient is the coefficient having $k-r+1$ as first index part and can not be eliminated. Therefore, only an elimination from (A1.4) or (A1.5) is possible each time. Specifically, l_1^b can be produced only from the inequality (A1.5), while any number of eliminations from (A1.4) can be applied, until (A1.5) occurs. Thus, l_1^b can be any number greater than $k-r+1$. Now, with the same argumentation as before, we conclude that l_2^b can be only greater than l_1^b. Continuing in a similar way, it results:

$$l_1^b \in \{k-r+2, ..., k\} \text{ and } l_j^b \in \left\{l_{j-1}^b + 1, ..., k\right\}, 2 \le j \le j_2.$$

Summarizing, all possible integer-sequences $\bar{l}$ that may appear in $P_{(k-r+1)\,\bar{l}\,(k+1)}$ are exactly those of the form $\bar{l} = \overline{l^a}\,\overline{l^b} \in \mathrm{N}^{j_1+j_2}$, with $\overline{l^a} = l_1^a l_2^a ... l_{j_1}^a \in \mathrm{N}^{j_1}$, $\overline{l^b} = l_1^b l_2^b ... l_{j_2}^b \in \mathrm{N}^{j_2}$, $j_1 \in \{0, 1, ..., k-r\}$, $j_2 \in \{0, 1, ..., r-1\}$, such that:

$$\begin{aligned}
&l_1^a \in \{1, ..., k-r\}, \quad l_1^b \in \{k-r+2, ..., k\}, \\
&l_j^a \in \left(\bigcap_{i=1}^{j-1} \{l_i^a - 1, ..., k-r\}\right) \setminus \left\{l_1^a, ..., l_{j-1}^a\right\}, for\ 2 \le j \le j_1, \\
&l_j^b \in \left\{l_{j-1}^b + 1, ..., k\right\}, for\ 2 \le j \le j_2.
\end{aligned}$$

5 Conclusions

The necessary and sufficient conditions for the existence of the solution of LMIs $\mathbf{A}(t)\,\mathbf{x} \ge \mathbf{b}(t), \forall t \in T$, where T is a finite, infinite, or even super countable set and $\mathbf{A}(t) \in \mathrm{R}^{N \times N}$ is a given triangular Toeplitz Matrix, have been presented. Also the restrictions of this solution, if such exists, have been derived using appropriate successive decompositions of the given inequalities into simpler ones. The above results may be extended in the more general case, where $\mathbf{A}(t) \in \mathrm{R}^{N \times N}$ is an arbitrary square matrix.

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Iterative Schemes with Mixed Errors for General Nonlinear Resolvent Operator Equations

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Abstract. In this paper, we introduce and study a new class of general nonlinear resolvent operator equations for maximal η-monotone mapping, and construct some new Mann and Ishikawa type iterative algorithms with mixed errors in Hilbert spaces. We also prove the existence of solutions for this kind of general nonlinear resolvent operator equations and the convergence of iterative sequences generated by the algorithms. Our results improve and generalize the corresponding results of [4], [10], [12] and [13].

Key Words: General nonlinear resolvent operator equation, maximal η-monotone mapping, Mann and Ishikawa type iteration, fixed point, existence and convergence

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1. INTRODUCTION AND PRELIMINARIES

Variational inequality theory is a very powerful tool of current mathematical technology and so variational inequalities have been extended and generalized in different directions by using novel and innovative techniques and ideas, both for their own sake and for their applications. In recent years, classical variational inequality problems have been extended and generalized to study a wide class of problems arising in optimization and control, economic and transportation equilibrium, and engineering sciences, etc., see [1]-[5], [8]-[10], [13] and the references therein.

Inspired and motivated by the recent works, in this paper, we introduce and study a new class of general nonlinear resolvent operator equations, and construct some new Mann and Ishikawa type iterative algorithms with mixed errors in Hilbert spaces. We also prove the existence of solutions for this class of general nonlinear resolvent operator equations and the convergence of iterative sequences generated by the algorithms. Our results improve and generalize the corresponding results of [4], [10], [12] and [13].

Throughout this paper, let X be a real Hilbert space which inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let k be a given positive integer, $h, g, T_i (i = 1, 2, \cdots, k) : X \to X$ be three nonlinear mappings and $M : X \to 2^X$ be a maximal η-monotone mapping, where 2^X denotes the family of nonempty subsets of X. For any constants $\rho_i > 0 (i = 1, 2, \cdots, k)$, we consider the problem of finding $x \in X$ such that

$$h(x) = J^M_{\rho_k}(g - \rho_k T_k)J^M_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots J^M_{\rho_1}(g - \rho_1 T_1)(x), \tag{1}$$

where $J^M_{\rho_i} = (I + \rho_i M)^{-1}$ is the resolvent operator associated with M for $i = 1, 2, \cdots, k$ and I is the identity mapping. Problem (1) is called the general nonlinear resolvent operator equation for maximal η-monotone mapping.

If $M = \triangle\phi$, where $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous and η-subdifferentiable proper functional on X, then problem (1) is equivalent to finding $x \in X$ such that

$$h(x) = J_{\rho_k}^{\triangle\phi}(g - \rho_k T_k) J_{\rho_{k-1}}^{\triangle\phi}(g - \rho_{k-1}T_{k-1}) \cdots J_{\rho_1}^{\triangle\phi}(g - \rho_1 T_1)(x), \tag{2}$$

which is called the general nonlinear resolvent operator equation.

It is easy to know that if $k = 1$, $T_1 = T$ and $h = g$, then problem (2) reduces to finding $x \in X$ such that

$$\langle T(x), \eta(y, g(x))\rangle + \phi(y) - \phi(g(x)) \geq 0, \quad \forall y \in X,$$

whose some special cases has been considered by many authors extensively (see, for example, [2], [4]-[6], [8]).

If $\eta(x, y) = x - y$ for all $x, y \in X$, $K : X \to 2^X$ is a given set-valued mapping such that each $K(x)$ is closed convex subset of X (or $K(x) = m(x) + K$, where $m : X \to X$ and K is a closed convex subset of X) and ϕ is the indicator function of K, that is

$$\phi(x) = \delta_K(x) = \begin{cases} 0, & \text{if} \quad x \in K, \\ +\infty, & \text{otherwise}, \end{cases}$$

then $J_{\rho_i}^{\triangle\phi} = P_{\rho_i}^K$ for $i = 1, 2, \cdots, k$, the projections of X onto K, and so problem (2) is equivalent to finding $x \in X$ such that

$$h(x) = P_{\rho_k}^K(g - \rho_k T_k) P_{\rho_{k-1}}^K(g - \rho_{k-1}T_{k-1}) \cdots P_{\rho_1}^K(g - \rho_1 T_1)(x), \tag{3}$$

which is called the general nonlinear projection operator equations. The problem (3) is introduced and studied by Zhao and Sun [14] when $k = 1$ and $\rho_1 = 1$.

In the sequel, we give some concepts and lemmas.

Definition 1 Let $g : X \to X$ be a single-valued mapping. A mapping $T : X \to X$ is said to be μ-strongly monotone with respect to g if there exists a constant $\mu > 0$ such that

$$\langle T(x) - T(y), g(x) - g(y)\rangle \geq \mu\|x - y\|^2, \quad \forall x, y \in X.$$

Definition 2 A mapping $\eta : X \times X \to X$ is said to be

(i) δ-strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle x - y, \eta(x, y)\rangle \geq \delta\|x - y\|^2, \quad \forall x, y \in X;$$

(ii) τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau\|x - y\|, \quad \forall x, y \in X.$$

Definition 3 ([7]) Let $\eta : X \times X \to X$ be a single-valued mapping. A mapping $A : X \to 2^X$ is said to be

(i) η-monotone if

$$\langle u - v, \eta(x, y)\rangle \geq 0, \quad \forall x, y \in X, u \in A(x), v \in A(y);$$

(ii) strictly η-monotone if

$$\langle u - v, \eta(x, y)\rangle \geq 0, \quad \forall x, y \in X, u \in A(x), v \in A(y),$$

and equality holds if and only if $x = y$;

(iii) strongly η-monotone if there exists a constant $r > 0$ such that

$$\langle u - v, \eta(x, y)\rangle \geq r\|x - y\|^2, \quad \forall x, y \in X, u \in A(x), v \in A(y);$$

(iv) maximal η-monotone if A is η-monotone and $(I + \lambda A)(X) = X$ for all (equivalently, for some) $\lambda > 0$.

Definition 4 If $A : X \to 2^X$ is a maximal η-monotone mapping, then for a constant $\rho > 0$, the operator J_ρ^M from X into X defined by

$$J_\rho^A(z) = (I + \rho A)^{-1}(z), \quad \forall z \in X,$$

is said to be the resolvent operator associated with A, where $\eta : X \times X \to X$ is a strictly monotone mapping.

Remark 1 If $\eta(x, y) = x - y$ for all $x, y \in X$, then (i)-(iv) of Definition 3 reduce to the classical definitions of monotonicity, strict monotonicity, strong monotonicity, and maximal monotonicity, respectively. At the same time, the resolvent operator J_ρ^A defined by Definition 4 reduces to the usual one for maximal monotone mapping.

Lemma 1 [6] Let $\eta : X \times X \to X$ be δ-strongly monotone and τ-Lipschitz continuous, and $A : X \to 2^X$ be a maximal η-monotone mapping. Then for any $\rho > 0$, the resolvent operator J_ρ^A for A is Lipschitz continuous with constant τ/δ, i.e.,

$$\|J_\rho^A(x) - J_\rho^A(y)\| \leq \frac{\tau}{\delta}\|x - y\|, \quad \forall x, y \in X.$$

Remark 2 If $\eta(x, y) = x - y$ for all $x, y \in X$, then Lemma 1 reduces to Lemma 2.2 in [1].

Definition 5 ([2]) Let $\eta : X \times X \to X$ be a single-valued mapping. A proper function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is said to be η-subdifferentiable at a point $x \in X$ if there exists a point $\xi \in X$ such that

$$\varphi(y) - \varphi(x) \geq \langle \xi, \eta(y, x)\rangle, \quad \forall y \in X,$$

where ξ is called an η-subgradient of φ at x. The set of all η-subgradients of φ at x is denoted by $\triangle\varphi(x)$, i.e., η-subdifferential of φ at x is the mapping $\triangle\varphi : X \to 2^X$ defined by

$$\triangle\varphi(x) = \{\xi \in X : \varphi(y) - \varphi(x) \geq \langle \xi, \eta(y, x)\rangle, \quad \forall y \in X\}.$$

Remark 3 If $\eta(y, x) = y - x, \forall y, x \in X$ and φ is a proper convex lower semicontinuous functional on X, then Definition 5 reduces to the usual definitions of subdifferential of a functional φ. Moreover, if φ is differentiable at $x \in X$ and satisfies

$$\varphi(x + \lambda\eta(y, x)) \leq \lambda\varphi(y) + (1 - \lambda)\varphi(x), \quad \forall y \in X, \lambda \in [0, 1],$$

then φ is η-subdifferentiable at x (see [11, p.424]).

Definition 6 ([14]) A functional $f : X\times X \to \mathbb{R}\cup\{+\infty\}$ is said to be 0-diagonally quasi-convex(in short, 0-DQCV) in y if for any finite set $\{x_1, x_2, \cdots, x_n\} \subset X$ and any $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, $\min_{1\leq i\leq n} f(x_i, y) \leq 0$.

Lemma 2 ([6]) Let $\eta : X \times X \to X$ be Lipschitz continuous and strongly monotone such that $\eta(x, y) = -\eta(y, x)$ for all $x, y \in X$, and for any given $s \in X$, the function $h(z, x) = \langle s - x, \eta(z, x)\rangle$ is 0-$DQCV$ in z. Let $\varphi : X \to \mathbb{R}\bigcup\{+\infty\}$ be a lower semicontinuous and η-subdifferetiable functional. Then the η-subdifferential $\triangle\varphi$ defined by Definition 5 is maximal η-monotone.

2. EXISTENCE THEOREMS

In the section, we shall give the existence result of solution for the resolvent operator equation (1) by using the Banach contractive mapping principle.

Theorem 1 Let X be a real Hilbert space, $h : X \to X$ α-strongly monotone and β-Lipschitz continuous, $g : X \to X$ ϵ-Lipschitz continuous, and $T_i : X \to X$ γ_i-strongly monotone with respect to g and σ_i-Lipschitz continuous, where $i = 1, 2, \cdots, k$. Let $\eta : X \times X \to X$ be δ-strongly monotone and τ-Lipschitz continuous, and $M : X \to 2^X$ be a maximal η-monotone mapping. Assume that for $i = 1, 2, \cdots, k$, there exist constants $\rho_i > 0$ such that

$$\begin{cases} \epsilon^2 - 2\rho_i\gamma_i + \rho_i^2\sigma_i^2 > 0, \quad i = 1, 2, \cdots, k, \\ \prod_{i=1}^{k}(\epsilon^2 - 2\rho_i\gamma_i + \rho_i^2\sigma_i^2) < (\frac{\delta}{\tau})^k(1 - \sqrt{1 - 2\alpha + \beta^2})^2, \end{cases} \tag{4}$$

where k is a given positive integer. Then there exists a unique $x \in X$ such that equation (1) holds, i.e., the general nonlinear resolvent operator equation (1) has a unique solution in X.

Proof. Let

$$F(x) = x + h(x) - Q_k(x),$$

where

$$\begin{aligned} Q_k(x) &= J_{\rho_k}^M(g - \rho_k T_k)[Q_{k-1}(x)] \\ &= J_{\rho_k}^M(g - \rho_k T_k)J_{\rho_{k-1}}^M(g - \rho_{k-1}T_{k-1}) \cdots J_{\rho_1}^M(g - \rho_1 T_1)(x). \end{aligned}$$

Now we prove F is contractive mapping. In fact, for any $x, y \in X$,

$$\|F(x) - F(y)\| \leq \|x - y - (h(x) - h(y))\| + \|Q_k(x) - Q_k(y)]\|. \tag{5}$$

Since h is α-strongly monotone and β-Lipschitz continuous, g is ϵ-Lipschitz continuous, and T is γ-strongly monotone with respect to g and σ-Lipschitz continuous, from Lemma 1, we have

$$\|x - y - (h(x) - h(y))\|^2 \leq (1 - 2\alpha + \beta^2)\|x - y\|^2, \tag{6}$$

$$\begin{aligned}
&\|Q_k(x) - Q_k(y)]\|^2 \\
&= \|J_{\rho_k}^M(g - \rho_k T_k)[Q_{k-1}(x)] - J_{\rho_k}^M(g - \rho_k T_k)[Q_{k-1}(y)]\|^2 \\
&\le \tau\delta^{-1}\|(g - \rho_k T_k)[Q_{k-1}(x)] - (g - \rho_k T)[Q_{k-1}(y)]\|^2 \\
&= \tau\delta^{-1}\|g[Q_{k-1}(x)] - g[Q_{k-1}(y)] - \rho_k\{T_k[Q_{k-1}(x)] - T_k[Q_{k-1}(y)]\}\|^2 \\
&= \tau\delta^{-1}\{\|g[Q_{k-1}(x)] - g[Q_{k-1}(y)]\|^2 + \rho_k^2\|T_k[Q_{k-1}(x)] - T_k[Q_{k-1}(y)]\|^2 \\
&\quad -2\rho_k\langle g[Q_{k-1}(x)] - g[Q_{k-1}(y), T_k[Q_{k-1}(x)] - T_k[Q_{k-1}(y)]\rangle \\
&\le \tau\delta^{-1}(\epsilon^2 - 2\rho_k\gamma_k + \rho_k^2\sigma_k^2)\|Q_{k-1}(x) - Q_{k-1}(y)\|^2 \\
&\le (\tau\delta^{-1})^2(\epsilon^2 - 2\rho_k\gamma_k + \rho_k^2\sigma_k^2) \\
&\quad \cdot(\epsilon^2 - 2\rho_{k-1}\gamma_{k-1} + \rho_{k-1}^2\sigma_{k-1}^2)\|Q_{k-2}(x) - Q_{k-2}(y)\|^2 \\
&\le \cdots \\
&\le (\tau\delta^{-1})^k\prod_{i=1}^{k}(\epsilon^2 - 2\rho_i\gamma_i + \rho_i^2\sigma_i^2)\|x - y\|^2,
\end{aligned} \tag{7}$$

for all $\rho_i > 0$ $(i = 1, 2, \cdots, k)$. From Eqs. (5)-(7), we have

$$\|F(x) - F(y)\| \le \theta\|x - y\|, \tag{8}$$

where

$$\theta = \sqrt{1 - 2\alpha + \beta^2} + (\tau\delta^{-1})^{\frac{k}{2}}\sqrt{\prod_{i=1}^{k}(\epsilon^2 - 2\rho_i\gamma_i + \rho_i^2\sigma_i^2)}.$$

It follows from condition (4) that $0 < \theta < 1$ and so $F : X \to X$ is a contractive mapping, which shows that F has a unique fixed point in X. This completes the proof.

If $M = \triangle\phi$, then from Lemma 2 and Theorem 1, we can get the following theorem.

Theorem 2 Let X, h, g and T_i $(i = 1, 2, \cdots, k)$ be the same as in Theorem 1, and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and η-subdifferentiable proper functional on X. Let $\eta : X \times X \to X$ be δ-strongly monotone and τ-Lipschitz continuous such that $\eta(x, y) = -\eta(y, x)$ for all $x, y \in X$, and for any given $s \in X$, the function $f(z, x) = \langle s - x, \eta(z, x)\rangle$ is 0-$DQCV$ in z. If condition (4) holds, then problem (2) has a unique solution in X.

If $\eta(x, y) = x - y$ for all $x, y \in X$ and ϕ is a indicator function of a closed convex set K in X, then from Theorem 2, we can get the following theorem.

Theorem 3 Let X, h, g and T_i $(i = 1, 2, \cdots, k)$ be the same as in Theorem 1, and $K : X \to 2^X$ be a set-valued mapping such that for each $x \in X$, $K(x)$ is nonempty closed and convex. If the following condition holds:

$$\epsilon^2 - 2\rho_i\gamma_i + \rho_i^2\sigma_i^2 > 0, \quad \prod_{i=1}^{k}(\epsilon^2 - 2\rho_i\gamma_i + \rho_i^2\sigma_i^2) < (1 - \sqrt{1 - 2\alpha + \beta^2})^2,$$

where k is the same as in (4), then problem (3) has a unique solution in X.

Remark 4 If $k = 1$ and there exists a constant $\rho > 0$ such that

$$\begin{cases} 2\alpha > \beta^2, \\ \epsilon^2 < \frac{\gamma^2}{\sigma^2} + \delta\tau^{-1}(1 - \sqrt{1 - 2\alpha + \beta^2})^2, \\ |\rho - \frac{\gamma}{\sigma^2}| < \frac{\sqrt{\gamma^2 - \sigma^2[\epsilon^2 - \delta\tau^{-1}(1 - \sqrt{1 - 2\alpha + \beta^2})^2]}}{\sigma^2}, \end{cases}$$

then condition (4) holds and the resolvent operator equation

$$h(x) = J_\rho^M (g - \rho T)(x)$$

exists a unique solution $x \in X$.

3. ITERATIVE ALGORITHMS AND CONVERGENCE

In this section, we construct some new perturbed iterative algorithms with mixed errors for solving the general nonlinear resolvent operator equations and prove the convergence of the iterative sequence generated by the algorithm in Hilbert spaces.

Lemma 3 ([9]) Let $\{a_n\}, \{b_n\}, \{c_n\}$ be there sequences of nonnegative numbers satisfying the following conditions: there exists n_0 such that

$$a_{n+1} \le (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \ge n_0,$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0 (n \to \infty)$.

Lemma 4 ([15]) If $H(K_n, K) \to 0$ as $n \to \infty$, where $K_n, K \subset X$ and $H(\cdot, \cdot)$ is the Hausdorff metric on the family of all nonempty bounded closed subsets of X, then

$$\lim_{n\to\infty} \|P_\rho^{K_n}(z) - P_\rho^K(z)\| = 0, \quad \forall z \in X.$$

Algorithm 1 Let $F : X \to X$ be a nonlinear mapping. Then, for any given $x_0 \in X$, we have the following perturbed iterative scheme $\{x_n\}$:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n F(y_n) + \alpha_n u_n + w_n, \\ y_n = (1 - \beta_n)x_n + \beta_n F(x_n) + v_n, \\ \quad n = 0, 1, \cdots, \end{cases} \tag{9}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[0, 1]$, $\{u_n\}, \{v_n\}, \{w_n\}$ are three sequences in X satisfying the following conditions:

(i) $u_n = u'_n + u''_n$;

(ii) $\lim_{n\to\infty} \|u'_n\| = \lim_{n\to\infty} \|v_n\| = 0$;

(iii) $\sum_{n=0}^{\infty} \|u''_n\| < \infty$, $\sum_{n=0}^{\infty} \|w_n\| < \infty$.

If $\beta_n \equiv v_n \equiv 0$, then Algorithm 1 becomes to the following algorithm.

Algorithm 2 For any given $x_0 \in X$, we have the following perturbed iterative scheme $\{x_n\}$ with mixed errors:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n F(x_n) + \alpha_n u_n + w_n$$

for $n = 0, 1, \cdots$, where $\{\alpha_n\}, \{u_n\}$ and $\{w_n\}$ are the same as in Algorithm 1.

Algorithm 3 Suppose that $K_n, K \subset X$ are closed convex subsets for $n = 0, 1, \cdots$, and $h, g, T_i\,(i = 1, 2, \cdots, k) : X \to X$. For given $x_0 \in X$, the perturbed iterative scheme $\{x_n\}$ with mixed errors is defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n[y_n + h(y_n) \\ \qquad -P_{\rho_k}^{K_n}(g-\rho_k T_k)P_{\rho_{k-1}}^{K_n}(g-\rho_{k-1}T_{k-1})\cdots P_{\rho_1}^{K_n}(g-\rho_1 T_1)(y_n)] \\ \qquad +\alpha_n u_n + w_n, \\ y_n = (1-\beta_n)x_n + \beta_n[x_n + h(x_n) \\ \qquad -P_{\rho_k}^{K_n}(g-\rho_k T_k)P_{\rho_{k-1}}^{K_n}(g-\rho_{k-1}T_{k-1})\cdots P_{\rho_1}^{K_n}(g-\rho_1 T_1)(x_n)] + v_n, \\ \; n = 0, 1, \cdots, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[0, 1]$, $\{u_n\}, \{v_n\}, \{w_n\}$ are three sequences of the element of X introduced to take into account possible inexact computation such that the following conditions hold:

$$\text{(i)}\, u_n = u_n' + u_n''; \quad \text{(ii)} \lim_{n\to\infty} \|u_n'\| = \lim_{n\to\infty} \|v_n\| = 0;$$
$$\text{(iii)} \sum_{n=0}^{\infty} \|u_n''\| < \infty, \; \sum_{n=0}^{\infty} \|w_n\| < \infty.$$

Theorem 4 Let $X, h, g, T_i\,(i = 1, 2, \cdots, k), \eta$ and M be the same as in Theorem 1 and $F : X \to X$ be defined by

$$F(x) = x + h(x) - Q_k(x), \tag{10}$$

where $Q_k(x) = J_{\rho_k}^M(g-\rho_k T_k)[Q_{k-1}(x)] = J_{\rho_k}^M(g-\rho_k T_k)J_{\rho_{k-1}}^M(g-\rho_{k-1}T_{k-1})\cdots J_{\rho_1}^M(g-\rho_1 T_1)(x)$. Assume that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and there exist constants $\rho_i > 0\,(i = 1, 2, \cdots, k)$ such that condition (4) holds. Then the iterative sequences $\{x_n\}$ with mixed errors defined by equation (9) converge strongly to the unique solution x^* of problem (1).

Proof. It follows from Theorem 1 that there exists a unique $x^* \in X$ which is a solution of the general nonlinear resolvent operator equation (1) and so $h(x^*) = Q_k(x^*)$, i.e.,

$$x^* = F(x^*). \tag{11}$$

From Eqs. (9) and (11), we have

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &= \|(1-\alpha_n)x_n + \alpha_n F(y_n) + \alpha_n u_n + w_n - (1-\alpha_n)x^* - \alpha_n F(x^*)\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|F(y_n) - F(x^*)\| \\ &\quad +\alpha_n\|u_n'\| + (\|u_n''\| + \|\omega_n\|). \end{aligned} \tag{12}$$

It follows from the definition of F in equation (10) and the proof of (8) in Theorem 1 that

$$\begin{aligned} &\|F(y_n) - F(x^*)\| \\ &\le \|y_n - x^* - (h(y_n) - h(x^*))\| + \|Q_k(y_n) - Q_k(x^*)]\| \\ &\le \vartheta\|y_n - x^*\|, \end{aligned} \tag{13}$$

where

$$\vartheta = \sqrt{1-2\alpha+\beta^2} + (\tau\delta^{-1})^{\frac{k}{2}} \sqrt{\prod_{i=1}^{k}(\epsilon^2 - 2\rho_i\gamma_i + \rho_i^2\sigma_i^2)}.$$

From Eqs. (12) and (13), we have

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\vartheta\|y_n - x^*\| + \alpha_n\|u_n'\| + (\|u_n''\| + \|\omega_n\|). \end{aligned} \quad (14)$$

Similarly, we have

$$\|y_n - x^*\| \le (1-\beta_n+\beta_n\vartheta)\|x_n - x^*\| + \|v_n\|. \quad (15)$$

It follows from Eqs. (14) and (15) that

$$\begin{aligned} \|x_{n+1} - x^*\| \quad \le \quad & [1-\alpha_n(1-\vartheta(1-\beta_n+\vartheta\beta_n))]\|x_n - x^*\| \\ & +\alpha_n(\|u_n'\| + \vartheta\|v_n\|) + (\|u_n''\| + \|\omega_n\|). \end{aligned} \quad (16)$$

Since $\vartheta < 1$ and $0 < \beta_n \le 1 (n \ge 0)$, we have $1-\beta_n+\vartheta\beta_n < 1$ and $1-\vartheta(1-\beta_n+\vartheta\beta_n) > 1-\vartheta > 0$. Hence, from equation (16), we get

$$\begin{aligned} \|x_{n+1} - x^*\| \quad \le \quad & [1-\alpha_n(1-\vartheta)]\|x_n - x^*\| \\ & +\alpha_n(1-\vartheta)\cdot\frac{1}{1-\vartheta}(\|u_n'\| + \vartheta\|v_n\|) + (\|u_n''\| + \|\omega_n\|). \end{aligned} \quad (17)$$

It follows from $\sum\limits_{n=0}^{\infty}\alpha_n = \infty$, Lemma 3 and equation (17) that $\|x_n - x^*\| \to 0 (n\to\infty)$, i.e.,

$$x_n \to x^*.$$

This completes the proof.

Theorem 5 Let $X, h, g, T_i\,(i = 1, 2, \cdots, k)$ and F be the same as in Theorem 4. Suppose that $\sum\limits_{n=0}^{\infty}\alpha_n = \infty$ and there exist constants $\rho_i > 0\,(i = 1, 2, \cdots, k)$ such that condition (4) holds. Then the iterative sequences $\{x_n\}$ with mixed errors defined by Algorithm 2 converge strongly to the unique solution x^* of problem (1).

Remark 6 If $M = \triangle\phi$, then we have the corresponding results of Theorems 4 and 5 with respect to problem (2), respectively. Our results improve and generalize the corresponding results of [10] and [13].

Theorem 6 Suppose that $K_n, K \subset X$ are closed convex subsets for $n = 0, 1, \cdots$, such that $H(K_n, K) \to 0$ as $n \to \infty$. Let $h : X \to X$ be α-strongly monotone and β-Lipschitz continuous, $g : X \to X$ be ϵ-Lipschitz continuous, and $T_i : X \to X$ be γ_i-strongly monotone with respect to g and σ_i-Lipschitz continuous for all $i = 1, 2, \cdots, k$. If

$$\begin{cases} \epsilon^2 - 2\rho_i\gamma_i + \rho_i^2\sigma_i^2 > 0, \quad \text{for} \quad i = 1, 2, \cdots, k \\ \prod\limits_{i=1}^{k}(\epsilon^2 - 2\rho_i\gamma_i + \rho_i^2\sigma_i^2) < (1-\sqrt{1-2\alpha+\beta^2})^2, \end{cases} \quad (18)$$

then the sequence $\{x_n\}$ generated by Algorithm 3 converges to the unique x^* of problem (3).

Proof. It follows from Theorem 3 that there exists $x^* \in X$ which is a solution of the general nonlinear projection operator equation (3) and so

$$h(x^*) = P^K_{\rho_k}(g - \rho_k T_k)P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*). \tag{19}$$

From Algorithm 3 and equation (19), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \|(1-\alpha_n)x_n + \alpha_n[y_n + h(y_n) \\
&\quad -P^{K_n}_{\rho_k}(g - \rho_k T_k)P^{K_n}_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^{K_n}_{\rho_1}(g - \rho_1 T_1)(y_n)] \\
&\quad +\alpha_n u_n + w_n - (1-\alpha_n)x^* - \alpha_n[x^* - h(x^*) \\
&\quad +P^K_{\rho_k}(g - \rho_k T_k)P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*)]\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - x^* - (h(y_n) - h(x^*))\| \\
&\quad +\alpha_n\|P^{K_n}_{\rho_k}(g - \rho_k T_k)P^{K_n}_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^{K_n}_{\rho_1}(g - \rho_1 T_1)(y_n) \\
&\quad -P^K_{\rho_k}(g - \rho_k T_k)P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*)\| \\
&\quad +\alpha_n\|u'_n\| + (\|u''_n\| + \|\omega_n\|).
\end{aligned} \tag{20}$$

Since h is α-strongly monotone and β-Lipschitz continuous, g is ϵ-Lipschitz continuous, and T_i is γ_i-strongly monotone with respect to g and σ_i-Lipschitz continuous, we have

$$\|y_n - x^* - (h(y_n) - h(x^*))\|^2 \le (1 - 2\alpha + \beta^2)\|y_n - x^*\|^2, \tag{21}$$

$$\begin{aligned}
&\|P^{K_n}_{\rho_k}(g - \rho_k T_k)P^{K_n}_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^{K_n}_{\rho_1}(g - \rho_1 T_1)(y_n) \\
&\quad -P^K_{\rho_k}(g - \rho_k T_k)P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*)\| \\
&\le \|P^{K_n}_{\rho_k}(g - \rho_k T_k)P^{K_n}_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^{K_n}_{\rho_1}(g - \rho_1 T_1)(y_n) \\
&\quad -P^{K_n}_{\rho_k}(g - \rho_k T_k)P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*)\| \\
&\quad +\|P^{K_n}_{\rho_k}(g - \rho_k T_k)P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*)\| \\
&\quad -P^K_{\rho_k}(g - \rho_k T_k)P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*)\| \\
&\le \|(g - \rho_k T_k)P^{K_n}_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1})P^{K_n}_{\rho_{k-2}}(g - \rho_{k-2}T_{k-2}) \\
&\quad \cdots P^{K_n}_{\rho_1}(g - \rho_1 T_1)(y_n) - (g - \rho_k T_k)P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \\
&\quad \cdot P^K_{\rho_{k-2}}(g - \rho_{k-2}T_{k-2}) \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*)\| \\
&\quad +\|P^{K_n}_{\rho_k}(g - \rho_k T_k)P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*)\| \\
&\quad -P^K_{\rho_k}(g - \rho_k T_k)P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*)\| \\
&\le \delta_k\|P^{K_n}_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1})P^{K_n}_{\rho_{k-2}}(g - \rho_{k-2}T_{k-2}) \cdots P^{K_n}_{\rho_1}(g - \rho_1 T_1)(y_n) \\
&\quad -P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1})P^K_{\rho_{k-2}}(g - \rho_{k-2}T_{k-2}) \\
&\quad \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*)\| + b_k,
\end{aligned} \tag{22}$$

where

$$\delta_k = \sqrt{\epsilon^2 - 2\rho_k\gamma_k + \rho_k^2\sigma_k^2},$$

$$\begin{aligned}
b_k \quad = \quad &\|P^{K_n}_{\rho_k}(g - \rho_k T_k)P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*)\| \\
&-P^K_{\rho_k}(g - \rho_k T_k)P^K_{\rho_{k-1}}(g - \rho_{k-1}T_{k-1}) \cdots P^K_{\rho_1}(g - \rho_1 T_1)(x^*)\|,
\end{aligned}$$

$$
\begin{aligned}
&\|P^{K_n}_{\rho_{k-1}}(g-\rho_{k-1}T_{k-1})P^{K_n}_{\rho_{k-2}}(g-\rho_{k-2}T_{k-2})\cdots P^{K_n}_{\rho_1}(g-\rho_1T_1)(y_n)\\
&\quad -P^{K}_{\rho_{k-1}}(g-\rho_{k-1}T_{k-1})P^{K}_{\rho_{k-2}}(g-\rho_{k-2}T_{k-2})\cdots P^{K}_{\rho_1}(g-\rho_1T_1)(x^*)\|\\
&\le \delta_{k-1}\|P^{K_n}_{\rho_{k-2}}(g-\rho_{k-2}T_{k-2})P^{K_n}_{\rho_{k-3}}(g-\rho_{k-3}T_{k-3})\cdots P^{K_n}_{\rho_1}(g-\rho_1T_1)(y_n)\\
&\quad -P^{K}_{\rho_{k-2}}(g-\rho_{k-2}T_{k-2})P^{K}_{\rho_{k-3}}(g-\rho_{k-3}T_{k-3})\cdots P^{K}_{\rho_1}(g-\rho_1T_1)(x^*)\|+b_{k-1}\\
&\le \delta_{k-1}\{\delta_{k-2}\|P^{K_n}_{\rho_{k-2}}(g-\rho_{k-2}T_{k-2})P^{K_n}_{\rho_{k-3}}(g-\rho_{k-3}T_{k-3})\cdots P^{K_n}_{\rho_1}(g-\rho_1T_1)(y_n)\\
&\quad -P^{K}_{\rho_{k-2}}(g-\rho_{k-2}T_{k-2})P^{K}_{\rho_{k-3}}(g-\rho_{k-3}T_{k-3})\\
&\quad \cdots P^{K}_{\rho_1}(g-\rho_1T_1)(x^*)\|+b_{k-2}\}+b_{k-1}\\
&\le \cdots\\
&\le \delta_{k-1}\cdots\delta_2\delta_1\|y_n-x^*\|\\
&\quad +\delta_{k-1}\delta_{k-2}\cdots\delta_2 b_1+\cdots+\delta_{k-1}\delta_{k-2}b_{k-3}+\delta_{k-1}b_{k-2}+b_{k-1},
\end{aligned}
\tag{23}
$$

and for $i=1,2,\cdots,k-1$,

$$
\begin{aligned}
\delta_i &= \sqrt{\epsilon^2-2\rho_i\gamma_i+\rho_k^2\sigma_i^2},\\
b_i &= \|P^{K_n}_{\rho_i}(g-\rho_iT_i)P^{K}_{\rho_{i-1}}(g-\rho_{i-1}T_{i-1})\cdots P^{K}_{\rho_1}(g-\rho_1T_1)(x^*)\|\\
&\quad -P^{K}_{\rho_i}(g-\rho_iT_i)P^{K}_{\rho_{i-1}}(g-\rho_{i-1}T_{i-1})\cdots P^{K}_{\rho_1}(g-\rho_1T_1)(x^*)\|.
\end{aligned}
$$

From Eqs. (20)-(23), it follows that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|\\
&\le (1-\alpha_n)\|x_n-x^*\|+\alpha_n(\sqrt{1-2\alpha+\beta^2}+\prod_{i=1}^k\delta_i)\|y_n-x^*\|\\
&\quad +\alpha_n(\prod_{i=2}^k\delta_i b_1+\prod_{i=3}^k\delta_i b_2+\cdots+\delta_k\delta_{k-1}b_{k-2}+\delta_k b_{k-1}+b_k)\\
&\quad +\alpha_n\|u_n'\|+(\|u_n''\|+\|\omega_n\|)\\
&= (1-\alpha_n)\|x_n-x^*\|+h\alpha_n\|y_n-x^*\|\\
&\quad +\alpha_n(s_n+\|u_n'\|)+(\|u_n''\|+\|\omega_n\|),
\end{aligned}
\tag{24}
$$

where

$$
h=\sqrt{1-2\alpha+\beta^2}+\prod_{i=1}^k\delta_i,\quad s_n=\sum_{j=1}^{k-1}\prod_{i=j+1}^k\delta_i b_j+b_k.
$$

Similarly, we have

$$
\|y_n-x^*\|\le(1-\beta_n)\|x_n-x^*\|+h\beta_n\|x_n-x^*\|+\beta_n s_n+\|v_n\|. \tag{25}
$$

It follows from Eqs. (24) and (25) that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|\\
&\le [1-\alpha_n+h\alpha_n(1-\beta_n+h\beta_n)]\|x_n-x^*\|\\
&\quad +h\alpha_n(\beta_n s_n+\|v_n\|)+\alpha_n(s_n+\|u_n'\|)+(\|u_n''\|+\|\omega_n\|).
\end{aligned}
\tag{26}
$$

From Eqs. (18) and (26), we know that $0<h<1$ and so

$$
\begin{aligned}
&\|x_{n+1}-x^*\|\\
&\le [1-\alpha_n(1-h)]\|x_n-x^*\|\\
&\quad +\alpha_n(1-h)\frac{(1+h\beta_n)s_n+h\|v_n\|+\|u_n'\|}{1-h}+(\|u_n''\|+\|\omega_n\|).
\end{aligned}
\tag{27}
$$

Using Lemma 4, we know that $\lim_{n\to\infty} b_i \to 0$ for all $i = 1, 2, \cdots, k$ and so $s_n \to 0$ as $n \to \infty$. It follows from $\sum_{n=0}^{\infty} \alpha_n = \infty$, Lemma 3 and equation (27) that $x_n \to x^*$. This completes the proof.

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Eigenfunctions of a class of Fredholm-Stieltjes integral equations via the inverse iteration method

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Abstract

We use the inverse iteration method in order to approximate the first eigenvalues and eigenfunctions of Fredholm-Stieltjes integral equations of second kind, i.e. Fredholm equations with respect to suitable Stieltjes-type measures. Some applications to the problem of a string with varying density $\rho(x)$ and charged by a finite number of cursors are given.

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1 Introduction

Some interesting articles of Carlo Miranda [9], [10] are dedicated to suitable extensions of the Fredholm theory. Namely, he studied the distribution of eigenvalues of the so called *charged Fredholm integral equations*, i.e. Fredholm equations

$$\int_I K(x,y)\varphi(y)dM_y = \mu\varphi(x) \tag{1.1}$$

with respect to a Stieltjes measure

$$dM_y = dy + \sum_{h=1}^{r} m_h\delta(a_h), \tag{1.2}$$

obtained by adding to the ordinary Lebesgue measure a finite combination of positive masses m_h concentrated in arbitrary points a_h $(h = 1, 2, \ldots, r)$ belonging to the interval I (here δ denotes the Dirac-Delta measure). In equation (1.1) $K(x, y) = K(y, x)$ denotes a strictly positive symmetric kernel.

We recall that in the uni-dimensional case this problem has an important mechanical interpretation, in fact it is connected with the consideration of free vibrations of a string charged by a finite number of cursors (see: [3], [13], [15], [6]).

Although at present the results of Miranda are included as a particular case in the theory of positive compact operators, they are of course the most important ones by the point of view of applications.

We already used the so called *inverse iteration method* performing some numerical experiments in order to approximate the eigenvalues of a second kind Fredholm operator [12], and even the eigenvalues in the case studied by C. Miranda [11].

Here we consider the general case of a string with a varying density $\rho(x)$ eventually charged by a finite number of cursors, so that the relevant measure will be given by

$$dM_y = \rho(y)dy + \sum_{h=1}^{r} m_h \delta(a_h).$$

So let L^2_{dM} be the Hilbert space equipped by the scalar product

$$[u, v]_{dM} := (u, v)_{dx_\rho} + (u, v)_{\Delta_r(x)}\,, \tag{1.3}$$

where

$$(u, v)_{dx_\rho} := \int_I u(x)v(x)\rho(x)dx \tag{1.4}$$

and

$$(u, v)_{\Delta_r(x)} := \sum_{h=1}^{r} m_h u(a_h)v(a_h). \tag{1.5}$$

It is worth noting that L^2_{dM} is constituted by functions of the wheighted $L^2_{\rho(x)}$, space, a (complete) Hilbert space, which does not have singularities at points a_h $(h = 1, 2, \ldots, r)$, according to the usual condition for existence of the relevant Stieltjes integral (see e.g. [16]).

Consider in L^2_{dM} the eigenvalue problem

$$\mathcal{K}\varphi = \mu\varphi, \tag{1.6}$$

where $\mathcal{K} : L^2_{dM}(I) \to L^2_{dM}(I)$ is the compact operator defined by

$$\mathcal{K}\varphi := \int_I K(\cdot, y)\varphi(y)dM_y \tag{1.7}$$

with symmetric kernel and Stieltjes measure defined by

$$dM_y = \rho(y)dy + \sum_{h=1}^{r} m_h \delta(a_h). \tag{1.8}$$

According to the above positions we have

$$\begin{aligned}(\mathcal{K}\varphi)(x) &= \int_I K(x,y)\varphi(y)\rho(y)dy + \sum_{h=1}^{r} m_h K(x,a_h)\varphi(a_h) = \\ &= (K(x,y),\varphi(y))_{dy_\rho} + (K(x,y),\varphi(y))_{\Delta_r(y)},\end{aligned}$$

so that

$$\begin{aligned}[\mathcal{K}u,v]_{dM} &= ((\mathcal{K}u)(x),v(x))_{dx_\rho} + ((\mathcal{K}u)(x),v(x))_{\Delta_r(x)} = \\ &= \Big((K(x,y),u(y))_{dy_\rho} + (K(x,y),u(y))_{\Delta_r(y)}, v(x)\Big)_{dx_\rho} + \\ &+ \Big((K(x,y),u(y))_{dy_\rho} + (K(x,y),u(y))_{\Delta_r(y)}, v(x)\Big)_{\Delta_r(x)},\end{aligned}$$

i.e.

$$\begin{aligned}&[\mathcal{K}u,v]_{dM} = \\ &= \Big((K(x,y),u(y))_{dy_\rho}, v(x)\Big)_{dx_\rho} + \Big((K(x,y),u(y))_{\Delta_r(y)}, v(x)\Big)_{dx_\rho} + \\ &+ \Big((K(x,y),u(y))_{dy_\rho}, v(x)\Big)_{\Delta_r(x)} + \Big((K(x,y),u(y))_{\Delta_r(y)}, v(x)\Big)_{\Delta_r(x)}.\end{aligned}$$

Furthermore

$$\Big((K(x,y),u(y))_{dy_\rho}, v(x)\Big)_{dx_\rho} = \int\int_{I\times I} K(x,y)u(y)v(x)\rho(x)\rho(y)dxdy,$$

$$\Big((K(x,y),u(y))_{\Delta_r(y)}, v(x)\Big)_{dx_\rho} = \sum_{h=1}^{r} m_h\ u(a_h) \int_I K(x,a_h)v(x)\rho(x)dx,$$

$$\Big((K(x,y),u(y))_{dy_\rho}, v(x)\Big)_{\Delta_r(x)} = \sum_{h=1}^{r} m_h\ v(a_h) \int_I K(x,a_h)u(x)\rho(x)dx,$$

where we used the symmetry of the kernel, and lastly

$$\Big((K(x,y),u(y))_{\Delta_r(y)}, v(x)\Big)_{\Delta_r(x)} = \sum_{h=1}^{r}\sum_{j=1}^{r} m_h\ m_j K(a_h,a_j)u(a_h)v(a_j).$$

After briefly recalling, in the present case, the inverse iteration method we develop some numerical examples about the approximation of the first eigenvalues.

2 The inverse iteration method

This method provides, in generally, an improvement of the lower bounds of eigenvalues obtained by means of the Rayleigh-Ritz method [8], [4].

In the following, for the sake of simplicity, we will limit ourselves to the case when I is an interval of the real axis, assuming $I = [0, 1]$, but the same technique could be used in the multidimensional case.

Furthermore, we suppose the kernel $K(x, y)$ to be a continuous function in the square $Q := [0, 1] \times [0, 1]$, or to admit, at most, a finite number of singular points inside Q.

By the generalization of Fredholm theorems proved by C. Miranda, the second kind homogeneous Fredholm integral equation (1.6) admits at most a denumerable set of non vanishing characteristic values which does not accumulate to finite points. Moreover, considering that by hypothesis our kernel is real, symmetric and strictly positive, the eigenvalues are real and positive so that they can be ordered in the following way

$$0 < \ldots \leq \mu_3 \leq \mu_2 \leq \mu_1. \tag{2.1}$$

We limit ourselves to this last case, since this is the most important case in physical applications.

It is worth noting that assuming that the kernel $K(x, y)$ is sufficiently regular and strictly positive with respect to the weighted Lebesgue $\rho(x)dx$ measure, it is possible to prove that it is strictly positive too with respect to the Lebesgue-Stieltjes measure dM. This follows from the series expansion of the kernel in terms of his eigenfunctions: $K(x, y) \simeq \sum_{k=1}^{\infty} \mu_k \phi_k(x)\phi_k(y)$. Assuming that the regularity of the considered kernel implies the point-wise convergence of the above expansion, it turns out that

$$[\mathcal{K}u, u]_{dM} = \sum_{k=1}^{\infty} c_k \left(\int_I \phi_k(x)u(x)\rho(x)dx + \sum_{h=1}^{r} m_h \phi_k(a_h)u(a_h) \right)^2 \geq 0 \ ,$$

and the last equality holds if and only if $u(x) \equiv 0$.

Assuming $j \geq 2$, suppose we know an initial approximation $\tilde{\mu}$ of the searched eigenvalue μ_j, such that

$$|\tilde{\mu} - \mu_j| < \frac{1}{2} \left(\min_{\mu_k \neq \mu_j, \, (k=1,2,\ldots,\nu)} |\mu_k - \mu_j| \right) \ ,$$

for a suitable choice of the integer ν. In practice in this condition the eigenvalues will be replaced by their Raileigh-Ritz approximations, for sufficiently large ν:

$$|\tilde{\mu} - \mu_j^{(\nu)}| < \frac{1}{2} \left(\min_{\mu_k^{(\nu)} \neq \mu_j^{(\nu)}, \, (k=1,2,\ldots,\nu)} |\mu_k^{(\nu)} - \mu_j^{(\nu)}| \right) \ . \tag{2.2}$$

From (1.6) we get:

$$(\mathcal{K} - \tilde{\mu}\mathcal{I})\varphi = (\mu - \tilde{\mu})\varphi, \tag{2.3}$$

where $\mathcal{I}$ denote the identity operator.

Consequently, if μ_j is an eigenvalue of $\mathcal{K}$ with eigenfunction φ_j, then $\mu_j - \tilde{\mu}$ is an eigenvalue of $\mathcal{K} - \tilde{\mu}\mathcal{I}$ with eigenfunction φ_j. By writing (2.3) in the form

$$(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}\varphi = (\mu - \tilde{\mu})^{-1}\varphi \tag{2.4}$$

it follows that $(\mu_j - \tilde{\mu})^{-1}$ is an eigenvalue of $(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}$ with the same eigenfunction φ_j.

By using condition (2.2), for ν sufficiently large, the eigenvalue $(\mu_j - \tilde{\mu})^{-1}$ becomes the (unique) eigenvalue of maximum modulus for the operator $(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}$. This leads to the possibility to apply the Kellog method (see [8]) in order to approximate $(\mu_j - \tilde{\mu})^{-1}$, and a corresponding eigenfunction. This can be done in the usual way, starting from an arbitrary function ω_0 (which theoretically should not be orthogonal to the eigenspace associated with $(\mu_j - \tilde{\mu})^{-1}$), and defining the sequence

$$\omega_{n+1} := (\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}\omega_n, \qquad (n = 0, 1, 2, \ldots).$$

Then (see [8]):

$$\lim_{n\to\infty} \frac{||\omega_{n+1}||_{L^2_{dM}}}{||\omega_n||_{L^2_{dM}}} = (\mu_j - \tilde{\mu})^{-1},$$

$$\lim_{n\to\infty} \frac{\omega_{2n}}{||\omega_{2n}||_{L^2_{dM}}} = \pm\varphi_j.$$

After computing with prescribed accuracy the eigenvalue

$$\xi_j := (\mu_j - \tilde{\mu})^{-1},$$

one finds

$$\mu_j = \frac{1}{\xi_j} + \tilde{\mu}.$$

It is important to note that (as in the finite dimensional case) we can avoid the determination of the inverse operator $(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}$, since the equation

$$\omega_{n+1} = (\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}\omega_n$$

is equivalent to

$$(\mathcal{K} - \tilde{\mu}\mathcal{I})\omega_{n+1} = \omega_n, \tag{2.5}$$

which in the case under examination becomes:

$$\int_0^1 (K(x,y)-\tilde{\mu})\,\rho(y)\omega_{n+1}(y)dy+\sum_{h=1}^{r} m_h\,(K(x,a_h)-\tilde{\mu})\,\omega_{n+1}(a_h)=\omega_n(x). \qquad (2.6)$$

However this leads to the necessity to solve numerically, at each step, a Fredholm-Stieltjes integral equation of the first kind.

This can be done by using different methods (see [2]-[1]), namely we could use, e.g., a suitable adaptation of the Fast Galerkin method, or of the Nyström method. The latter method was used, since it turned out to be very simple and efficient both with respect to time and number of iterations.

The rate of convergence of the method is given by the formula:

$$\frac{||\omega_n||_{L^2_{dM}}}{||\omega_0||_{L^2_{dM}}}=\mathcal{O}\left[(\mu'(\mu_j-\tilde{\mu}))^n\right],$$

where $\mu'\neq(\mu_j-\tilde{\mu})^{-1}$ denotes a suitable eigenvalue of $(\mathcal{K}-\tilde{\mu}\mathcal{I})^{-1}$ (see [14]).

As a matter of fact, by the numerical point of view, the use of Nyström method in the solution of equation (2.6) is substantially equivalent to the substitution of the original kernel $K(x,y)$ by an approximating kernel $\tilde{K}(x,y)$ given by a suitably defined two-dimensional step function (i.e. instead of the original operator, we consider an approximating finite dimensional operator given by a suitable matrix). In order to define this finite dimensional operator, and to discuss the accuracy of our approximation we introduce some notations.

Suppose, first, for fixing our attention, that the points $a_1,a_2,\ldots,a_r$ are enumerated in increasing order, and do not coincide with the extrema of the interval $[0,1]$. This condition is not essential, but in the opposite case, suitable modifications of the following definitions are necessary.

Let

$$0=:a_0<a_1<\ldots<a_r<a_{r+1}:=1,$$

and in each interval $[a_h,a_{h+1}]$ consider the m nodes of the modified Gauss-Legendre quadrature rule, $x_1^{(h)},x_2^{(h)},\ldots,x_m^{(h)}$ putting

$$a_h=:x_0^{(h)}<x_1^{(h)}<\ldots<x_m^{(h)}<x_{m+1}^{(h)}:=a_{h+1}, \qquad (0\le h\le r).$$

The corresponding points, when belonging to the y-axis, will be denoted as:

$$a_k=:y_0^{(k)}<y_1^{(k)}<\ldots<y_m^{(k)}<y_{m+1}^{(k)}:=a_{k+1}, \qquad (0\le k\le r).$$

The relevant Christoffel constants will be denoted by $w_1^{(k)},w_2^{(k)},\ldots,w_m^{(k)}$.

Consider the subintervals $Q_{i,j}^{(h,k)}$ of Q defined by

$$Q_{i,j}^{(h,k)}:=\{(x,y)\mid x_i^{(h)}<x<x_{i+1}^{(h)};\ y_j^{(k)}<y<y_{j+1}^{(k)}\},$$
$$(i,j=0,1,\ldots,m;\ h,k=0,1,\ldots,r)$$

and by $\hat{Q}_{i,j}^{(h,k)}$ those among the preceding ones, for which the closure $\overline{Q}_{i,j}^{(h,k)}$ contains at least a singular point of the kernel $K(x,y)$.

Then define

$$\tilde{K}(x,y) = \begin{cases} K(x_i,y_j), & \text{if } (x,y) \in Q_{i,j}^{(h,k)} \\ \\ M_{i,j}, & \text{if } (x,y) \in \hat{Q}_{i,j}^{(h,k)} \end{cases} \tag{2.7}$$

where $M_{i,j}$ are constants such that

$$||K(x,y) - \tilde{K}(x,y)||_{L^2_{dM}(\cup_{i,j;h,k}\ \hat{Q}_{i,j}^{(h,k)})} < eps,$$

where eps denotes the smallest positive number used by the computer (i.e. the *machine epsilon*). This condition can always be satisfied provided that m is sufficiently large.

As a consequence the application of the Nyström method consists in the solution of the algebraic system of order $m(r+1)+r$ obtained by assuming $x = x_\ell^{(h)}$ $(\ell = 1,2,\ldots,m;\ h = 0,1,\ldots,r)$ and $x = a_i$ $(i = 1,2,\ldots,r)$ into the following equation:

$$\sum_{k=0}^{r}\sum_{s=1}^{m} w_s^{(k)} \left(\tilde{K}(x,y_s^{(k)}) - \tilde{\mu}\right) \rho(y_s^{(k)})\omega_{n+1}(y_s^{(k)}) +$$
$$+\sum_{j=1}^{r} m_j \left(\tilde{K}(x,a_j) - \tilde{\mu}\right) \omega_{n+1}(a_j) = \omega_n(x).$$

Consequently assuming that $\omega_n(x)$ is known, the subsequent term $\omega_{n+1}(x)$ of the above mentioned sequence is given by the Lagrange interpolating polynomial

$$\omega_{n+1}(x) = \sum_{h=0}^{r}\sum_{\ell=1}^{m} L_{\ell,h}(x)\omega_{n+1}(x_\ell^{(h)}) + \sum_{i=1}^{r} L_i(x)\omega_{n+1}(a_i)$$

where the $L_{\ell,h}$ and L_i are the basic Lagrange polynomials corresponding to the given nodes $x_\ell^{(h)}$ $(\ell = 1,2,\ldots,m;\ h = 0,1,\ldots,r)$ and to the points a_i $(i = 1,\ldots,r)$, whose number is exactly $m(r+1)+r$.

Then the numerical computation by using the inverse power method yields to approximate the exact eigenvalues $\tilde{\mu}_j$, $(j = 1,2,\ldots,\nu)$ of the kernel $\tilde{K}(x,y)$. Anyway, by using the well known Aronszajn Theorem (see e.g. [5]), it is possible to find an upper bound for the absolute error $|\mu_j - \tilde{\mu}_j|$, which is given simply, for every j, by the estimate

$$|\mu_j - \tilde{\mu}_j| \leq ||K(x,y) - \tilde{K}(x,y)||_{L^2_{dM}(Q)}.$$

Then, in order to find an approximation $\tilde{\mu}_j$ which is exact, with respect to the corresponding μ_j, up to the p^{th} digit, it is sufficient to increase ν (and eventually to

use adaptive composite quadrature formulas, increasing the number of knots close to the singularities) in such a way that the further inequality $||K(x,y) - \tilde{K}(x,y)||_{L^2_{dM}(Q)} < .5 \times 10^{-p}$ holds true.

This can always be done, and permits to control the error of our approximation, independently of the orthogonal invariants method.

3 Numerical examples

In each of the following examples we use two programs written, by second author, by means of the computer algebra system Mathematica © to obtain the approximations of the first six eigenvalues and related eigenfunctions about problems of type (1.6)-(1.7)-(1.8), where

$$K(x,y) = \begin{cases} x(1-y) & \text{if } 0 \le x \le y \le 1 \\ \\ y(1-x) & \text{if } 0 \le y \le x \le 1. \end{cases} \tag{3.1}$$

The considered operator is compact and strictly positive, since it deals with the free vibrations of a string charged by a finite number of cursors, and the mechanical interpretation of $[\mathcal{K}u, u]_{dM}$ is twice the deformation work performed by u (see e.g. [7], p. 364). This work is always strictly positive, unless $u \equiv 0$.

In particular the first program implements the Rayleigh-Ritz method while the second one implements the inverse iteration method. We denote with $\tilde{\mu}_k$ and $\tilde{\varphi}_k$ respectively the lower bounds of the eigenvalues and the approximations of the eigenfunctions obtained by means the first program while with $\hat{\mu}_k$ and $\hat{\varphi}_k$ the improvement, obtained by means the second program, of the above approximations. Moreover, in both programs, we compute the norms

$$\tilde{n}_k := ||\tilde{\Omega}_k||_{L^2_{dM}} \qquad \text{and} \qquad \hat{n}_k := ||\hat{\Omega}_k||_{L^2_{dM}}$$

where

$$\tilde{\Omega}_k = (\mathcal{K} - \tilde{\mu}_k \mathcal{I})\tilde{\varphi}_k \qquad \text{and} \qquad \hat{\Omega}_k = (\mathcal{K} - \hat{\mu}_k \mathcal{I})\hat{\varphi}_k.$$

Example 1.

Let $dM_y = \rho(y)dy$ where $\rho(y) = 1 + y$. We obtain the following approximated values:

$\tilde{\mu}_1 = 0.1527091780202889$	$\tilde{n}_1 = 1.6492663 \times 10^{-3}$
$\tilde{\mu}_2 = 0.03778580471524793$	$\tilde{n}_2 = 3.209913837 \times 10^{-3}$
$\tilde{\mu}_3 = 0.01675759185142281$	$\tilde{n}_3 = 4.952451994 \times 10^{-3}$
$\tilde{\mu}_4 = 0.00941879544238094$	$\tilde{n}_4 = 6.420799629 \times 10^{-3}$
$\tilde{\mu}_5 = 0.00602577862779985$	$\tilde{n}_5 = 8.254744931 \times 10^{-3}$
$\tilde{\mu}_6 = 0.004183725497809093$	$\tilde{n}_6 = 9.632083719 \times 10^{-3}$

Rayleigh-Ritz method

$\hat{\mu}_1 = 0.1527095935266794$	$\hat{n}_1 = 1.269055735 \times 10^{-15}$
$\hat{\mu}_2 = 0.03778619450648527$	$\hat{n}_2 = 1.416080348 \times 10^{-14}$
$\hat{\mu}_3 = 0.01675800399601823$	$\hat{n}_3 = 1.371525918 \times 10^{-13}$
$\hat{\mu}_4 = 0.00941918561140838$	$\hat{n}_4 = 4.766983115 \times 10^{-13}$
$\hat{\mu}_5 = 0.006026192402056793$	$\hat{n}_5 = 1.056268286 \times 10^{-12}$
$\hat{\mu}_6 = 0.004184117877808688$	$\hat{n}_6 = 7.513709468 \times 10^{-13}$

Inverse iteration method

Remark 3.1 *This example was formerly considered by S.G. Mikhlin (see* [8]*, pag. 277–280). By means of a suitable method he gives the following lower bounds for* $\lambda_1 = \frac{1}{\mu_1}$

$$\frac{1}{\sqrt[4]{A_4}} < \lambda_1 < \frac{\sqrt{A_2}}{\sqrt{A_4}}$$

and the following approximation for $\lambda_2 = \frac{1}{\mu_2}$

$$\lambda_2 \approx \frac{1}{\lambda_1}\sqrt{\frac{2}{A_2^2 - A_4}},$$

where $A_2 = \frac{127}{5040}$ *and* $A_4 = 0.0006154$ *so that he found* $6.349 < \lambda_1 < 6.398$ *and* $\lambda_2 \approx 49.9$. *Consequently it results:*

$$0.156298843 < \mu_1 < 0.157505118 \quad \textit{and} \quad \mu_2 \approx 0.02004008.$$

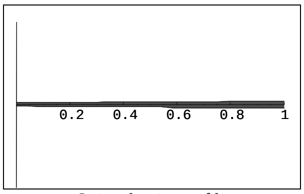
However we checked a mistake in the value of A_4 *(pag. 279 of* [8]*). In fact, by using the computer algebra software Mathematica* ©*, we found*

$$A_4 = \frac{5949413}{10897286400} \approx 0.000545954.$$

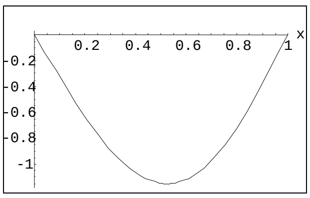
Consequently, the correct values are:

$$0.147194545 < \mu_1 < 0.152858219 \quad \textit{and} \quad \mu_2 \approx 0.043642203.$$

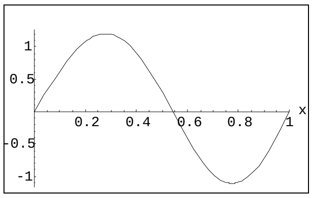
Graphs of the approximate eigenfunctions are shown in Fig. 1.



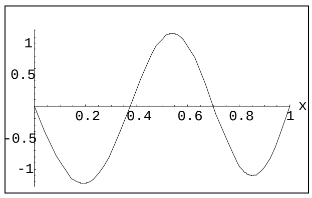
String density profile



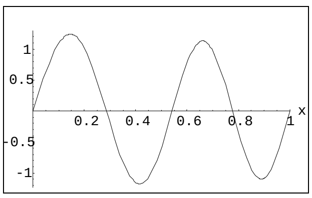
Approximate eigenfunction $\hat{\varphi}_1$



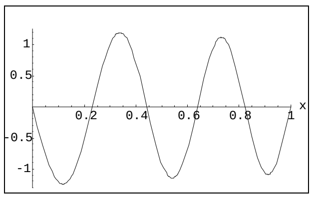
Approximate eigenfunction $\hat{\varphi}_2$



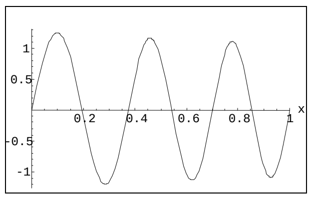
Approximate eigenfunction $\hat{\varphi}_3$



Approximate eigenfunction $\hat{\varphi}_4$



Approximate eigenfunction $\hat{\varphi}_5$



Approximate eigenfunction $\hat{\varphi}_6$

Example 2.

Let $dM_y = \rho(y)dy$ where $\rho(y) = \sin(\pi y)$. We obtain the following approximated values:

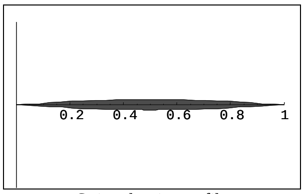
$$
\begin{array}{ll}
\tilde{\mu}_1 = 0.0864220486915503 & \tilde{n}_1 = 2.48507115 \times 10^{-4} \\
\tilde{\mu}_2 = 0.01764748388197546 & \tilde{n}_2 = 4.55649292 \times 10^{-4} \\
\tilde{\mu}_3 = 0.007366070626598713 & \tilde{n}_3 = 6.808030359 \times 10^{-4} \\
\tilde{\mu}_4 = 0.004019269688238463 & \tilde{n}_4 = 8.387956573 \times 10^{-4} \\
\tilde{\mu}_5 = 0.002526645954564795 & \tilde{n}_5 = 1.05941064 \times 10^{-3} \\
\tilde{\mu}_6 = 0.001734033750570599 & \tilde{n}_6 = 1.187371598 \times 10^{-3}
\end{array}
$$

Rayleigh-Ritz method

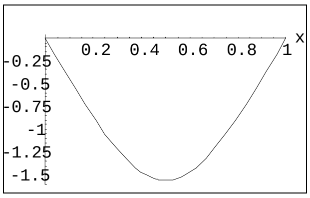
$$
\begin{array}{ll}
\hat{\mu}_1 = 0.0864220540288647 & \hat{n}_1 = 1.527106229 \times 10^{-15} \\
\hat{\mu}_2 = 0.01764748754668348 & \hat{n}_2 = 3.783905457 \times 10^{-15} \\
\hat{\mu}_3 = 0.007366074042615985 & \hat{n}_3 = 1.190945686 \times 10^{-12} \\
\hat{\mu}_4 = 0.004019272518830757 & \hat{n}_4 = 3.700433721 \times 10^{-14} \\
\hat{\mu}_5 = 0.002526648794942703 & \hat{n}_5 = 4.337030741 \times 10^{-12} \\
\hat{\mu}_6 = 0.001734036200757312 & \hat{n}_6 = 6.425896357 \times 10^{-13}
\end{array}
$$

Inverse iteration method

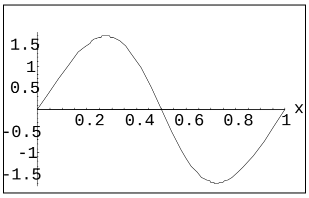
Graphs of the approximate eigenfunctions are shown in Fig. 2.



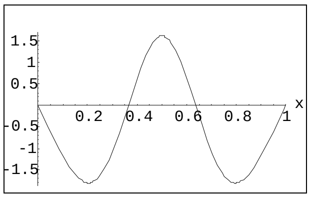
String density profile



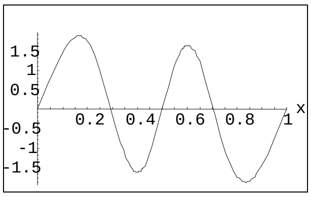
Approximate eigenfunction $\hat{\varphi}_1$



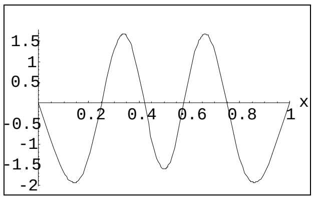
Approximate eigenfunction $\hat{\varphi}_2$



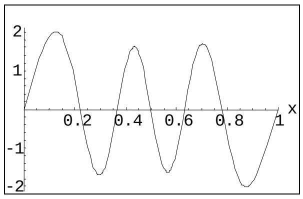
Approximate eigenfunction $\hat{\varphi}_3$



Approximate eigenfunction $\hat{\varphi}_4$



Approximate eigenfunction $\hat{\varphi}_5$



Approximate eigenfunction $\hat{\varphi}_6$

Example 3.

Let $dM_y = \rho(y)dy + \sum_{h=1}^{3} m_h \delta(a_h)$ where $\rho(y) = 1 - \cos(8\pi y)$, $a_h = \frac{h}{4}$ $(h = 1, 2, 3)$ and $m_1 = m_3 = \frac{1}{2}$, $m_2 = 1$. We obtain the following approximated values:

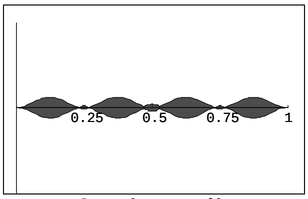
$\tilde{\mu}_1 = 0.4206426870614815$	$\tilde{n}_1 = 2.38490472 \times 10^{-4}$
$\tilde{\mu}_2 = 0.0821759584821933$	$\tilde{n}_2 = 8.105483947 \times 10^{-4}$
$\tilde{\mu}_3 = 0.05504780650729236$	$\tilde{n}_3 = 1.493964885 \times 10^{-3}$
$\tilde{\mu}_4 = 0.0096239654631904$	$\tilde{n}_4 = 6.726581587 \times 10^{-4}$
$\tilde{\mu}_5 = 0.00915527459333726$	$\tilde{n}_5 = 2.628488603 \times 10^{-3}$
$\tilde{\mu}_6 = 0.007667921028682865$	$\tilde{n}_6 = 4.554111005 \times 10^{-3}$

Rayleigh-Ritz method

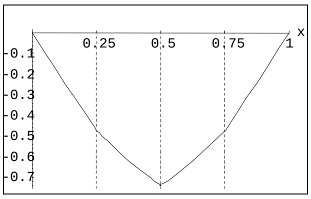
$\hat{\mu}_1 = 0.4206427109875892$	$\hat{n}_1 = 1.926263045 \times 10^{-15}$
$\hat{\mu}_2 = 0.0821760124809624$	$\hat{n}_2 = 3.143745418 \times 10^{-14}$
$\hat{\mu}_3 = 0.0550479294048313$	$\hat{n}_3 = 5.583870374 \times 10^{-14}$
$\hat{\mu}_4 = 0.00962396982280946$	$\hat{n}_4 = 4.666455934 \times 10^{-13}$
$\hat{\mu}_5 = 0.00915533795244373$	$\hat{n}_5 = 8.256812178 \times 10^{-12}$
$\hat{\mu}_6 = 0.007668080386305962$	$\hat{n}_6 = 1.356411863 \times 10^{-12}$

Inverse iteration method

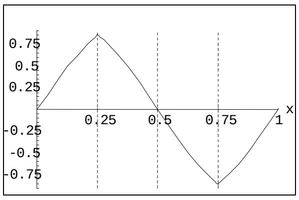
Graphs of the approximate eigenfunctions are shown in Fig. 3.



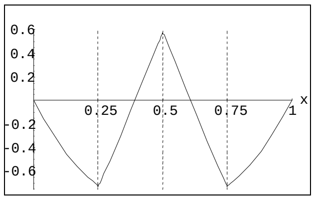
String density profile



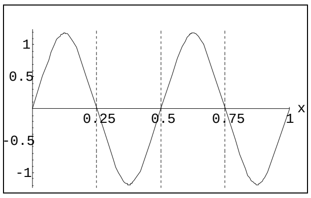
Approximate eigenfunction $\hat{\varphi}_1$



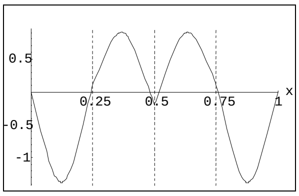
Approximate eigenfunction $\hat{\varphi}_2$



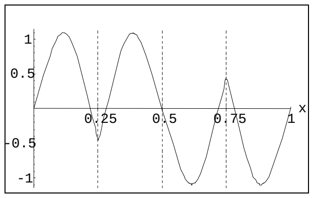
Approximate eigenfunction $\hat{\varphi}_3$



Approximate eigenfunction $\hat{\varphi}_4$



Approximate eigenfunction $\hat{\varphi}_5$



Approximate eigenfunction $\hat{\varphi}_6$

Example 4.

Let $dM_y = \rho(y)dy + \sum_{h=1}^{4} m_h \delta(a_h)$ where $\rho(y) = 1 + \cos(8\pi y)$, $a_h = \frac{2h+1}{8}$ $(h = 1, \ldots, 4)$ and $m_1 = m_4 = \frac{1}{4}$, $m_2 = m_3 = \frac{1}{2}$. We obtain the following approximated values:

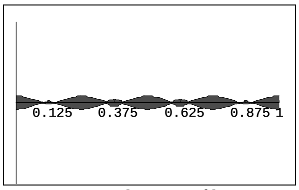
$\tilde{\mu}_1 = 0.29471968884813322$	$\tilde{n}_1 = 1.138055993 \times 10^{-3}$
$\tilde{\mu}_2 = 0.06972725939479091$	$\tilde{n}_2 = 2.127491416 \times 10^{-3}$
$\tilde{\mu}_3 = 0.02650081330382946$	$\tilde{n}_3 = 4.698707196 \times 10^{-3}$
$\tilde{\mu}_4 = 0.0232594968507961$	$\tilde{n}_4 = 4.734472141 \times 10^{-3}$
$\tilde{\mu}_5 = 0.00830294554513302$	$\tilde{n}_5 = 3.898650164 \times 10^{-3}$
$\tilde{\mu}_6 = 0.006872136484805615$	$\tilde{n}_6 = 5.893771572 \times 10^{-3}$

Rayleigh-Ritz method

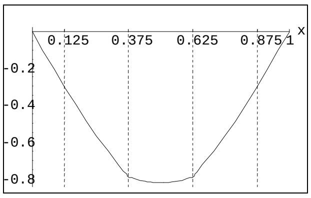
$\hat{\mu}_1 = 0.294720070244869$	$\hat{n}_1 = 6.890440822 \times 10^{-16}$
$\hat{\mu}_2 = 0.06972757515321075$	$\hat{n}_2 = 1.629453016 \times 10^{-14}$
$\hat{\mu}_3 = 0.02650139918380914$	$\hat{n}_3 = 1.039209604 \times 10^{-13}$
$\hat{\mu}_4 = 0.02326001893242146$	$\hat{n}_4 = 1.415656594 \times 10^{-14}$
$\hat{\mu}_5 = 0.00830307211766979$	$\hat{n}_5 = 3.291106196 \times 10^{-12}$
$\hat{\mu}_6 = 0.006872375998663793$	$\hat{n}_6 = 7.993863397 \times 10^{-13}$

Inverse iteration method

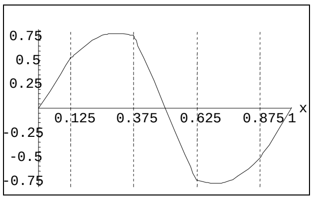
Graphs of the approximate eigenfunctions are shown in Fig. 4.



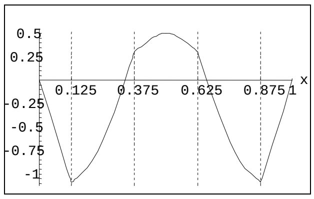
String density profile



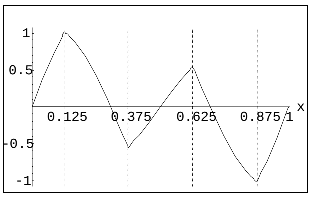
Approximate eigenfunction $\hat{\varphi}_1$



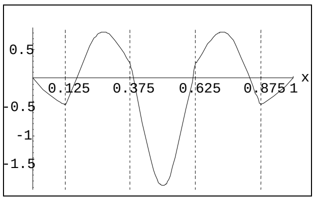
Approximate eigenfunction $\hat{\varphi}_2$



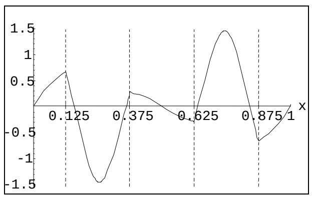
Approximate eigenfunction $\hat{\varphi}_3$



Approximate eigenfunction $\hat{\varphi}_4$



Approximate eigenfunction $\hat{\varphi}_5$



Approximate eigenfunction $\hat{\varphi}_6$

Example 5.

Let $dM_y = \rho(y)dy + \sum_{h=1}^{4} m_h \delta(a_h)$ where

$$\rho(y) = \begin{cases} 16y^2 - 4y + \frac{1}{4} & \text{if } 0 \le y \le \frac{1}{4} \\ 16y^2 - 12y + \frac{9}{4} & \text{if } \frac{1}{4} \le y \le \frac{1}{2} \\ 16y^2 - 20y + \frac{25}{4} & \text{if } \frac{1}{2} \le y \le \frac{3}{4} \\ 16y^2 - 28y + \frac{49}{4} & \text{if } \frac{3}{4} \le y \le 1, \end{cases} \tag{3.2}$$

$a_h = \frac{2h-1}{8}$ $(h = 1, \dots, 4)$ and $m_1 = m_4 = \frac{1}{4}$, $m_2 = m_3 = \frac{1}{2}$. We obtain the following approximated values:

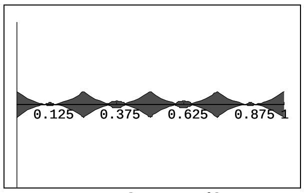
$\tilde{\mu}_1 = 0.2068296728995903$	$\tilde{n}_1 = 4.213949527 \times 10^{-4}$
$\tilde{\mu}_2 = 0.05254180101694188$	$\tilde{n}_2 = 6.83829442 \times 10^{-4}$
$\tilde{\mu}_3 = 0.02003649239417445$	$\tilde{n}_3 = 1.693081718 \times 10^{-3}$
$\tilde{\mu}_4 = 0.01936003246493382$	$\tilde{n}_4 = 1.634094291 \times 10^{-3}$
$\tilde{\mu}_5 = 0.000862709702997794$	$\tilde{n}_5 = 1.712615212 \times 10^{-3}$
$\tilde{\mu}_6 = 0.000845845848020365$	$\tilde{n}_6 = 2.946821985 \times 10^{-3}$

Rayleigh-Ritz method

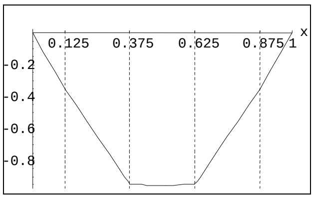
$\hat{\mu}_1 = 0.2068297096277673$	$\hat{n}_1 = 1.292520163 \times 10^{-15}$
$\hat{\mu}_2 = 0.05254182558814604$	$\hat{n}_2 = 6.508453848 \times 10^{-15}$
$\hat{\mu}_3 = 0.02003654983896096$	$\hat{n}_3 = 6.453394196 \times 10^{-15}$
$\hat{\mu}_4 = 0.01936008416976135$	$\hat{n}_4 = 1.010224718 \times 10^{-14}$
$\hat{\mu}_5 = 0.000862712236694869$	$\hat{n}_5 = 2.795312715 \times 10^{-11}$
$\hat{\mu}_6 = 0.000845853202527304$	$\hat{n}_6 = 3.833732973 \times 10^{-12}$

Inverse iteration method

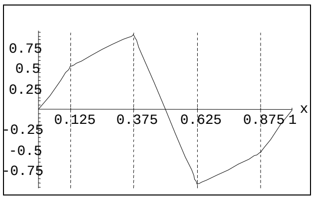
Graphs of the approximate eigenfunctions are shown in Fig. 5.



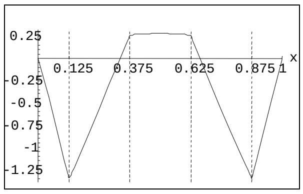
String density profile



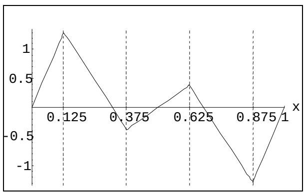
Approximate eigenfunction $\hat{\varphi}_1$



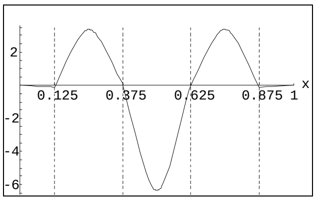
Approximate eigenfunction $\hat{\varphi}_2$



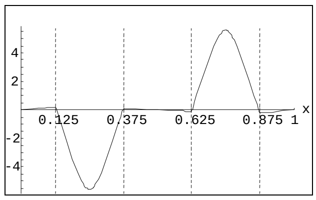
Approximate eigenfunction $\hat{\varphi}_3$



Approximate eigenfunction $\hat{\varphi}_4$



Approximate eigenfunction $\hat{\varphi}_5$



Approximate eigenfunction $\hat{\varphi}_6$

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Nonlinear A-Monotone Variational Inclusions Systems and the Resolvent Operator Technique

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Abstract

Based on the notion of the $A-$*monotonicity*, the solvability of a system of nonlinear variational inclusions using the resolvent operator technique is given. The obtained results are general in nature.

Mathematics Subject Classifications: 49J40, 65B05, 47J20

Key Words and Phrases: $A-$*monotonicity*, Variational inclusion systems, Resolvent operator technique, Maximal monotone mappings.

1. Introduction

The notion of the $A-$*monotonicity* [8] has applications to problems ranging from variational inclusion system problems to hemivariational inclusion problems, especially from engineering. The $A-$*monotonicity* in fact generalizes the $h-$*monotonicity*$-$ introduced by Fang and Huang [2] in the context of solving some nonlinear inclusion systems in Hilbert space settings. These notions do impact greatly the theory of maximal monotone mappings in terms of applications. Our variational inclusion systems generalize [2] to the case of $A-$*monotone* mappings in Hilbert spaces. As a matter of fact, the $A-$*monotonicity* stems from the work of Naniewicz and Panagiotopoulos [6] and Verma [12], where its special variants have been applied in solving the constrained hemivariational inequalities on reflexive Banach spaces. The author [12] investigated the solvability of some constrained nonlinear hemivariational inequality problems based on the technique of

generalized maximal monotone mappings by Naniewicz and Panagiotopoulos [6]. We note that the $A-monotonicity$ is represented in terms of relaxed monotone mappings - a more general notion than the monotonicity/strong monotonicity. In this paper, we consider the solvability of a system of nonlinear variational inclusions based on the resolvent operator technique. The obtained results are general in nature

Definition 1. [1] Let $h : H \to H$ and $M : H \to 2^H$ be any mappings on H. The map M is said to be $h-monotone$ if M is monotone and $(h + \rho M)(H) = H$ for $\rho > 0$, where H is a Hilbert space.

Note that if h is strictly monotone and M is $h-monotone$, then M is maximal monotone. Let the resolvent operator $J^{\rho}_{h,M}(u) = (h + \rho M)^{-1}(u)\ \forall u \in H$.

Definition 2. [8] Let $A : X \to X^*$ and $M : X \to 2^{X^*}$ be any mappings on X, where X is a reflexive Banach space and X^* its dual. The map M is said to be $A-monotone$ if M is $m - relaxed$ monotone and $(A + \rho M)$ is maximal monotone for $\rho > 0$.

Lemma 1. Let A :$H \to H$ be $r - strongly$ monotone and $M : H \to 2^H$ be A-monotone. Then the resolvent operator $J^{\rho}_{A,M} : H \to H$ is $(\frac{1}{r-\rho m}) - Lipschitz$ continuous for $0 < \rho < \frac{r}{m}$.

Proof. For any $u, v \in H$, we have from the definition of the resolvent operator that

$$J^{\rho}_{A,M}(u) = (A + \rho M)^{-1}(u)$$

$$J^{\rho}_{A,M}(v) = (A + \rho M)^{-1}(v).$$

It follows that

$$(\frac{1}{\rho})[u - A(J^{\rho}_{A,M}(u))] \in M(J^{\rho}_{A,M}(u))$$

$$(\frac{1}{\rho})[v - A(J^{\rho}_{A,M}(v))] \in M(J^{\rho}_{A,M}(v)).$$

Since M is $A-monotone$ (and hence, $m - relaxed$ monotone), it implies that

$$(\frac{1}{\rho})\langle[u - A(J^{\rho}_{A,M}(u))] - [v - A(J^{\rho}_{A,M}(v))], J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\rangle$$

$$= (\frac{1}{\rho})\langle u - v - [A(J^{\rho}_{A,M}(u)) - A(J^{\rho}_{A,M}(v))], J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\rangle$$

$$\geq (-m)\|J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\|^2.$$

As a result, we have

$$\begin{aligned}
& \|u-v\|\|J^{\rho}_{A,M}(u)-J^{\rho}_{A,M}(v)\| \\
\geq\ & \langle u-v, J^{\rho}_{A,M}(u)-J^{\rho}_{A,M}(v)\rangle \\
\geq\ & \langle A(J^{\rho}_{A,M}(u))-A(J^{\rho}_{A,M}(v)), J^{\rho}_{A,M}(u))-J^{\rho}_{A,M}(v))\rangle \\
-\ & \rho m\|J^{\rho}_{A,M}(u)-J^{\rho}_{A,M}(v)\|^2 \\
\geq\ & r\|J^{\rho}_{A,M}(u)-J^{\rho}_{A,M}(v)\|^2 \\
-\ & \rho m\|J^{\rho}_{A,M}(u)-J^{\rho}_{A,M}(v)\|^2 \\
=\ & (r-\rho m)\|J^{\rho}_{A,M}(u)-J^{\rho}_{A,M}(v)\|^2.
\end{aligned}$$

This completes the proof.

Example 1. [6, Lemma 7.11] Let $A : X \to X^*$ be $(m) - strongly$ monotone and $f : X \to R$ be locally Lipschitz such that ∂f is $(\alpha) - relaxed$ monotone. Then ∂f is $A-monotone$, that is, $A + \partial f$ is maximal monotone for $m - \alpha > 0$, where $m, \alpha > 0$.

Example 2. [12, Theorem 4.1] Let $A : X \to X^*$ be $(m) - strongly$ monotone and $B : X \to X^*$ be $(c) - strongly$ Lipschitz continuous. Let $f : X \to R$ be locally Lipschitz such that ∂f is $(\alpha) - relaxed$ monotone. Then ∂f is $(A - B)-monotone$.

Next, let H_1 and H_2 be two real Hilbert spaces. Let $M : H_1 \to 2^{H_1}$ and $N : H_2 \to 2^{H_2}$ be nonlinear mappings. Let $S : H_1 \times H_2 \to H_1$ and $T : H_1 \times H_2 \to H_2$ be any two mappings. Then the problem of finding $(a, b) \in H_1 \times H_2$ such that

$$0 \in S(a, b) + M(a), \tag{1}$$

$$0 \in T(a, b) + N(b), \tag{2}$$

is called the system of nonlinear variational inclusion (abbreviated SNVI) problems.

Lemma 2. Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_1$ and $B : H_2 \to H_2$ be strictly monotone, $M : H_1 \to 2^{H_1}$ be $A-monotone$ and $N : H_2 \to 2^{H_2}$ be $B-monotone$. Let $S : H_1 \times H_2 \to H_1$ and $T : H_1 \times H_2 \to H_2$ be any two multivalued mappings. Then a given element $(a, b) \in H_1 \times H_2$ is a solution to the $SNVI(1) - (2)$ problem iff (a, b) satisfies

$$a = J^{\rho}_{A,M}(A(a) - \rho S(a, b)), \tag{3}$$

$$b = J^{\eta}_{B,N}(B(b) - \eta T(a, b)), \tag{4}$$

where $\rho, \eta > 0$.

Theorem 1. Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_1$ be $(r_1)-strongly$ monotone and $(\alpha_1)-Lipschitz$ continuous, and let $B : H_2 \to H_2$ be $(r_2)-strongly$ monotone and $(\alpha_2)-Lipschitz$ continuous. Let $M : H_1 \to 2^{H_1}$ be $A-monotone$ and let $N : H_2 \to 2^{H_2}$ be $B-monotone$. Let $S : H_1 \times H_2 \to H_1$ be such that $S(.,y)$ is $(r)-strongly$ monotone and $(\mu)-Lipschitz$ continuous in the first variable and $S(x,.)$ is $(\nu)-Lipschitz$ continuous in the second variable for all $(x,y) \in H_1 \times H_2$. Let $T : H_1 \times H_2 \to H_2$ be such that $T(u,.)$ is $(s)-strongly$ monotone and $(\beta)-Lipschitz$ continuous in the second variable and $T(.,v)$ is $(\tau)-Lipschitz$ continuous in the first variable for all $(u,v) \in H_1 \times H_2$. If, in addition, there exist positive constants ρ and η such that

$$(r_2 - \eta p)\sqrt{\alpha_1^2 - 2\rho r + \rho^2\mu^2} + \eta\tau(r_1 - \rho m) < (r_1 - \rho m)(r_2 - \eta p)$$

$$(r_1 - \rho m)\sqrt{\alpha_2^2 - 2\eta s + \eta^2\beta^2} + (\rho\nu)(r_2 - \eta p) < (r_1 - \rho m)(r_2 - \eta p),$$

then the $SNVI(1)-(2)$ problem has a unique solution.

Here M and N, respectively, are $m-relaxed$ monotone and $p-relaxed$ monotone for $m, p > 0$.

Proof. Let us define mappings $S^*(a,b)$ and $T^*(a,b)$, respectively, by

$$S^*(a,b) = J^{\rho}_{A,M}[A(a) - \rho S(a,b)]$$

$$T^*(a,b) = J^{\eta}_{B,N}[B(b) - \eta T(a,b)].$$

Then for any elements $(u,v),(w,x) \in H_1 \times H_2$, we have from $Lemma$ 1 that

$$\begin{aligned}
&\|S^*(u,v) - S^*(w,x)\| \\
&\quad \leq \frac{1}{r_1 - \rho m}\|A(u) - A(w) - \rho[S(u,v) - S(w,v)]\| \\
&\quad + \frac{\rho}{r_1 - \rho m}\|S(w,v) - S(w,x)\| \\
&\quad \leq \frac{1}{r_1 - \rho m}\{\sqrt{\alpha_1^2 - 2\rho r + \rho^2\mu^2}\|u - w\| + \rho\nu\|v - x\|\}
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\|T^*(u,v) - T^*(w,x)\| \\
&\quad \leq \frac{1}{r_2 - \eta p}\|B(v) - B(x) - \eta[T(u,v) - T(u,x)]\| \\
&\quad + \frac{1}{r_2 - \eta p}\|T(u,x) - T(w,x)\| \\
&\quad \leq \frac{1}{r_2 - \eta p}\{\sqrt{\alpha_2^2 - 2\eta s + \eta^2\beta^2}\|v - x\| + \eta\tau\|u - w\|\}
\end{aligned}$$

It follows from the above arguments that

$$\begin{aligned}
& \|S^*(u,v)-S^*(w,x)\|+\|T^*(u,v)-T^*(w,x)\| \\
\leq\ & \frac{1}{r_1-\rho m}\{\sqrt{\alpha_1^2-2\rho r+\rho^2\mu^2}\|u-w\|+\rho\nu\|v-x\|\} \\
+\ & \frac{1}{r_2-\eta p}\{\sqrt{\alpha_2^2-2\eta s+\eta^2\beta^2}\|v-x\|+\eta\tau\|u-w\|\} \\
=\ & [\frac{1}{r_1-\rho m}\sqrt{\alpha_1^2-2\rho r+\rho^2\mu^2}+\frac{\eta\tau}{r_2-\eta p}]\|u-w\| \\
+\ & [\frac{1}{r_2-\eta p}\sqrt{\alpha_2^2-2\eta s+\eta^2\beta^2}+\frac{\rho\nu}{r_1-\rho m}]\|v-x\| \\
\leq\ & max\{[\frac{1}{r_1-\rho m}\sqrt{\alpha_1^2-2\rho r+\rho^2\mu^2}+\frac{\eta\tau}{r_2-\eta p}] \\
,\ & [\frac{1}{r_2-\eta p}\sqrt{\alpha_2^2-2\eta s+\eta^2\beta^2}+\frac{\rho\nu}{r_1-\rho m}]\}[\|u-w\|+\|v-x\|].
\end{aligned}$$

Set

$$\begin{aligned}
k\ =\ & max\{[\frac{1}{r_1-\rho m}\sqrt{\alpha_1^2-2\rho r+\rho^2\mu^2}+\frac{\eta\tau}{r_2-\eta p}] \\
,\ & [\frac{1}{r_2-\eta p}\sqrt{\alpha_2^2-2\eta s+\eta^2\beta^2}+\frac{\rho\nu}{r_1-\rho m}]\}.
\end{aligned}$$

Next, we define the norm $\|(u,v)\|^*$ by

$$\|(u,v)\|^*=\|u\|+\|v\|\forall(u,v)\in H_1\times H_2.$$

Clearly, $H_1\times H_2$ is a Banach space with the norm $\|(u,v)\|^*$. We define a mapping $U:H_1\times H_2\to H_1\times H_2$ by

$$U(u,v)=(S^*(u,v),T^*(u,v))\ \forall(u,v)\in H_1\times H_2.$$

Since $0<k<1$, it follows that

$$\begin{aligned}
& \|U(u,v)-U(w,x)\|^*=\|(S^*(u,v),T^*(u,v))-(S^*(w,x),T^*(w,x))\|^* \\
=\ & \|(S^*(u,v)-S^*(w,x)),(T^*(u,v)-T^*(w,x))\|^* \\
=\ & \|(S^*(u,v)-S^*(w,x)\|+\|(T^*(u,v)-T^*(w,x))\| \\
\leq\ & k[\|u-w\|+\|v-x\|] \\
=\ & k\|(u,v)-(w,x)\|^*.
\end{aligned}$$

Hence, U is a contraction. This implies that there exists a unique element $(a,b)\in H_1\times H_2$ such that

$$U(a,b)=(a,b),$$

which means,

$$a = J^{\rho}_{A,M}(A(a) - \rho S(a,b)),$$
$$b = J^{\eta}_{B,N}(B(b) - \eta T(a,b)),$$

where $\rho, \eta > 0$.

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On certain integral operators between weighted L^p spaces

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Abstract

In this paper we extend our previous results [3] concerning the behaviour of certain integral operators over weighted L^p spaces. Particular cases include the classical index transforms (cf. [9]) and the operators with complex Gaussian kernels (cf. [7], [8]).

Key words and phrases: weighted L^p inequalities, integral operators, index transforms, complex Gaussian kernels.

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1 Introduction

In a previous paper [3] we considered the integral operator given by

$$(\mathcal{F}f)(y) = \int_I f(x)K(x,y)dx, \tag{1.1}$$

where $y \in I$ (I denoting some interval in $\mathbb{R}$, possibly unbounded), over the spaces $L^p(I, w(x)dx), 1 \leq p < \infty$, under suitable conditions on the kernel K and the weight w.

We also considered the integral operator

$$(\mathcal{L}g)(x) = \int_I g(y)K(x,y)dy, \tag{1.2}$$

where I, w, and K are the same as in (1.1).

We proved some results concerning the boundedness of $\mathcal{L}$, Parseval relations containing $\mathcal{F}$ and $\mathcal{L}$, and we established that the adjoint operator of $\mathcal{L}$ can be viewed as a natural extension of the operator defined by (1.1).

These results hold whenever the following conditions are satisfied:

$$1 < p < \infty, p + q = pq,$$
$$w \geq 1 \ a.e. \ on \ I, \ \int_I w(y)^{-q/p} dy < \infty, \tag{1.3}$$
$$|K(x,y)| \leq \tilde{k}(x) \ a.e. \ on \ I, \ with \ \tilde{k} \in L^q(I, w(x)dx),$$

and for the case $p = 1$,

$$w \geq 1 \ a.e. \ on \ I, \ |K(x,y)| \leq \tilde{k}(x) \ a.e. \ on \ I, \tag{1.4}$$

and $\tilde{k}$ is essentially bounded on I.

The purpose of this paper is to prove these results with weaker conditions to (1.3) and (1.4) respectively. The condition (1.3) is replaced by the weaker one

$$1 < p < \infty, \ p + q = pq,$$
$$w \geq 1 \ a.e. \ on \ I, \ \int_I \int_I |K(x,y)|^q w(x) w(y)^{-q/p} dx dy < \infty, \tag{1.5}$$

and the condition (1.4) is replaced by the weaker one

$$p = 1, \ w \geq 1 \ a.e. \ on \ I,$$
$$ess \ sup_{x \in I} \ ess \ sup_{y \in I} \frac{|K(x,y)|}{w(y)} < \infty. \tag{1.6}$$

This new setting allow us to analyse operators with complex Gaussian kernels (cf. [7], [8]), which satisfy conditions (1.5) and (1.6). Observe that these Gaussians operators are not particular cases of the results of our previous paper [3].

Also, the main index transforms (cf. [9]) are included as particular cases of this general analysis.

As a novelty, we also study the case $p = \infty$ obtaining the results under the new conditions, namely,

$$p = \infty, \ w \geq 1 \ a.e. \ on \ I,$$
$$\int_I \int_I |K(x,y)| w(x) dx dy < \infty. \tag{1.7}$$

From now on, we denote by T_f the functional in $(L^p(I, w(x)dx))'$ given by $T_f(g) = \int_I f(x) g(x) dx$, for any $f \in L^p(I, w(x)dx)$.

Also $\mathcal{L}'$ denotes the adjoint of the operator $\mathcal{L}$, i.e.,

$$\langle \mathcal{L}' f, g \rangle = \langle f, \mathcal{L} g \rangle,$$

for any $f \in (L^q(I, w(x)dx))'$ and any $g \in L^p(I, w(x)dx)$.

2 The operator $\mathcal{L}$ on the spaces $L^p(I, w(x)dx), 1 < p < \infty$

In this section we study the behaviour of the operator $\mathcal{L}$ on the spaces $L^p(I, w(x)dx), 1 < p < \infty$.

Proposition 2.1. Assume $1 < p < \infty$, $p + q = pq$, $w \geq 1$ a.e. on I, and

$$\int_I \int_I |K(x,y)|^q w(x) w(y)^{-q/p} dx dy < \infty.$$

Then the operator $\mathcal{L}$ given by (1.2) is bounded from $L^p(I, w(x)dx)$ into $L^q(I, w(x)dx)$.

Proof. Let $g \in L^p(I, w(x)dx)$. From (1.2), we have

$$|(\mathcal{L}g)(x)| \leq \int_I |g(y)| w(y)^{1/p} |K(x,y)| w(y)^{-1/p} dy. \tag{2.1}$$

Using Hölder's inequality, the right-hand side of (2.1) is less than or equal to

$$\left(\int_I |g(y)|^p w(y) dy \right)^{1/p} \cdot \left(\int_I |K(x,y)|^q w(y)^{-q/p} dy \right)^{1/q}$$
$$= \left(\int_I |K(x,y)|^q w(y)^{-q/p} dy \right)^{1/q} \cdot \|g\|_p.$$

So, we have

$$\left(\int_I |(\mathcal{L}g)(x)|^q w(x) dx \right)^{1/q}$$
$$\leq \|g\|_p \cdot \left(\int_I \int_I |K(x,y)|^q w(x) w(y)^{-q/p} dx dy \right)^{1/q}.$$

From this inequality and taking into account the hypothesis, it follows that

$$\|\mathcal{L}g\|_q \leq M \cdot \|g\|_p,$$

for certain $M > 0$. □

Proposition 2.2. If $f \in L^p(I, w(x)dx)$, $g \in L^p(I, w(x)dx)$, $1 < p < \infty$, $p + q = pq$, $w \geq 1$ a.e. on I, and

$$\int_I \int_I |K(x,y)|^q w(x) w(y)^{-q/p} dx dy < \infty,$$

then the following Parseval-type relation holds

$$\int_I (\mathcal{F}f)(x) g(x) dx = \int_I f(x) (\mathcal{L}g)(x) dx. \tag{2.2}$$

Proof. For $f \in L^p(I, w(x)dx)$, we have

$$|(\mathcal{F}f)(x)| \leq \int_I |f(t)||K(t,x)|dt = \int_I |f(t)|w(t)^{1/p}|K(t,x)|w(t)^{-1/p}dt.$$

Using Hölder's inequality and the fact that $w \geq 1$ a.e. on I, we have

$$\begin{aligned}|(\mathcal{F}f)(x)| &\leq \left(\int_I |f(t)|^p w(t)dt\right)^{1/p} \cdot \left(\int_I |K(t,x)|^q w(t)^{-q/p}dt\right)^{1/q} \\ &\leq \|f\|_p \cdot \left(\int_I |K(t,x)|^q w(t)dt\right)^{1/q}.\end{aligned}$$

Therefore, using again Hölder's inequality, one has

$$\left|\int_I |(\mathcal{F}f)(x)||g(x)|dx \leq \|f\|_p \cdot \int_I \left(\int_I |K(t,x)|^q w(t)dt\right)^{1/q} \cdot |g(x)|dx\right|$$

$$\begin{aligned}&= \|f\|_p \cdot \int_I \left(\int_I |K(t,x)|^q w(t)dt\right)^{1/q} |g(x)|w(x)^{1/p}w(x)^{-1/p}dx \\ &\leq \|f\|_p \left[\int_I \int_I |K(t,x)|^q w(t)w(x)^{-q/p}dtdx\right]^{1/q} \cdot \left(\int_I |g(x)|^p w(x)dx\right)^{1/p} \\ &= \|f\|_p\|g\|_p \left[\int_I \int_I |K(t,x)|^q w(t)w(x)^{-q/p}dtdx\right]^{1/q} \\ &\leq M\|f\|_p\|g\|_p,\end{aligned}$$

for some $M > 0$.

On the other hand,

$$\begin{aligned}\int_I |f(x)||(\mathcal{L}g)(x)|dx &\leq \left(\int_I |f(x)|^p w(x)dx\right)^{1/p} \cdot \left(\int_I |(\mathcal{L}g)(x)|^q w(x)^{-q/p}dx\right)^{1/q} \\ &\leq \|f\|_p\|g\|_p \left(\int_I \int_I |K(x,y)|^q w(x)^{-q/p}w(y)^{-q/p}dxdy\right)^{1/q} \\ &\leq \|f\|_p\|g\|_p \left(\int_I \int_I |K(x,y)|^q w(x)w(y)^{-q/p}dxdy\right)^{1/q} \\ &\leq M\|f\|_p\|g\|_p,\end{aligned}$$

for some $M > 0$.

Finally, using Fubini's theorem, we obtain the Parseval-type formula given by (2.2). □

Now, the operator $\mathcal{L}'$ can be viewed as a natural extension of the operator defined by (1.1). In fact, given $f, g \in L^p(I, w(x)dx)$, we have

$$| < T_{\mathcal{F}f}, g > | \leq \|f\|_p \|g\|_p \left(\int_I \int_I |K(x,y)|^q w(x) w(y)^{-q/p} dx dy \right)^{1/q}.$$

So, from the hypothesis, it follows that $T_{\mathcal{F}f} \in (L^p(I, w(x)dx))'$.

On the other hand, for $f \in L^p(I, w(x)dx)$ and $g \in L^q(I, w(x)dx)$, we have

$$| < T_f, g > | \leq \int_I |f(x)||g(x)|dx \leq \int_I |f(x)||g(x)|w(x)dx,$$

whenever $w \geq 1$ a.e. on I. Since this last integral is less than or equal to $\|f\|_p \cdot \|g\|_q$, we conclude that $T_f \in (L^q(I, w(x)dx))'$.

Thus, we have proved the following result.

Corollary 2.1. If $f \in L^p(I, w(x)dx), 1 < p < \infty, p + q = pq$, with $w \geq 1$ a.e. on I, and

$$\int_I \int_I |K(x,y)|^q w(x) w(y)^{-q/p} dx dy < \infty,$$

then

$$\mathcal{L}' T_f = T_{\mathcal{F}f}$$

on $(L^p(I, w(x)dx))'$.

3 The operator $\mathcal{L}$ on the space $L^1(I, w(x)dx)$

We now prove corresponding results to Proposition 2.1 and Proposition 2.2 for the case when $p = 1$.

Proposition 3.1. Assume $w \geq 1$ a.e. on I and

$$ess\ sup_{x \in I}\ ess\ sup_{y \in I} \frac{|K(x,y)|}{w(y)} < \infty.$$

Then, the operator $\mathcal{L}$ given by (1.2) is bounded from $L^1(I, w(x)dx)$ into $L^\infty(I, w(x)dx)$.

Proof. Let $g \in L^1(I, w(x)dx)$. Then using Hölder's inequality one has

$$|(\mathcal{L}g)(x)| \leq \int_I |K(x,y)||g(y)|dy \leq ess\ sup_{y \in I} \frac{|K(x,y)|}{w(y)} \cdot \|g\|_1.$$

So

$$ess\ sup_{x \in I} |(\mathcal{L}g)(x)| \leq \|g\|_1 \cdot ess\ sup_{x \in I}\ ess\ sup_{y \in I} \frac{|K(x,y)|}{w(y)},$$

which completes the proof of Proposition 3.1. □

Proposition 3.2. If $f \in L^1(I, w(x)dx)$ and $g \in L^1(I, w(x)dx)$, $w \geq 1$ a.e. on I and

$$ess\ sup_{x\in I}\ ess\ sup_{y\in I} \frac{|K(x,y)|}{w(y)} < \infty,$$

then the following Parseval-type relation holds

$$\int_I (\mathcal{F}f)(x)g(x)dx = \int_I f(x)(\mathcal{L}g)(x)dx. \tag{3.1}$$

Proof. In fact,

$$|(\mathcal{F}f)(x)| \leq \int_I |f(t)||K(t,x)|dt \leq ess\ sup_{t\in I} \frac{|K(t,x)|}{w(t)} \cdot \|f\|_1.$$

Therefore

$$\int_I |(\mathcal{F}f)(x)||g(x)|dx \leq \|f\|_1 \cdot \int_I ess\ sup_{t\in I} \frac{|K(t,x)|}{w(t)} \cdot |g(x)|dx$$

$$\leq \|f\|_1 \cdot \|g\|_1 \cdot ess\ sup_{x\in I}\ ess\ sup_{t\in I} \frac{|K(t,x)|}{w(t)w(x)},$$

which from the hypothesis is less than or equal to

$$M\|f\|_1\|g\|_1$$

for some $M > 0$.

On the other hand,

$$\int_I |f(x)||(\mathcal{L}g)(x)|dx \leq M \cdot \|f\|_1 \cdot \|g\|_1,$$

for some $M > 0$.

Finally, using Fubini's theorem, we obtain (3.1). □

We now prove the connection between the operator $\mathcal{L}'$ and $\mathcal{F}$. Indeed, for $f, g \in L^1(I, w(x)dx)$, we have

$$| < T_{\mathcal{F}f}, g > | \leq \int_I |(\mathcal{F}f)(x)||g(x)|dx \leq M \cdot \|f\|_1 \cdot \|g\|_1,$$

for some $M > 0$. So $T_{\mathcal{F}f} \in (L^1(I, w(x)dx))'$.

On the other hand, for $f \in L^1(I, w(x)dx)$ and $g \in L^\infty(I, w(x)dx)$, we have

$$| < T_f, g > \leq \int_I |f(x)||g(x)|dx \leq \|g\|_\infty \cdot \|f\|_1,$$

from which we conclude that $T_f \in (L^\infty(I, w(x)dx))'$. Thus, we have proved the following result.

Corollary 3.1. If $f \in L^1(I, w(x)dx)$, $w \geq 1$ a.e. on I, and

$$ess\ sup_{x\in I}\ ess\ sup_{y\in I} \frac{|K(x,y)|}{w(y)} < \infty,$$

then

$$\mathcal{L}' T_f = T_{\mathcal{F}f}$$

on $(L^1(I, w(x)dx))'$.

4 The operator $\mathcal{L}$ on the space $L^\infty(I, w(x)dx)$

We also prove corresponding results to Proposition 2.1 and Proposition 2.2 for the case when $p = \infty$.

Proposition 4.1. Assume $w \geq 1$ a.e. on I and

$$\int_I \int_I |K(x,y)| w(x) dx dy < \infty.$$

Then the operator $\mathcal{L}$ given by (1.2) is bounded from $L^\infty(I, w(x)dx)$ into $L^1(I, w(x)dx)$.

Proof. Let $g \in L^\infty(I, w(x)dx)$. Then, using Hölder's inequality, we can write

$$|(\mathcal{L}g)(x)| \leq \int_I |K(x,y)||g(y)| dy$$

$$\leq ess\ sup_{y\in I} |g(y)| \cdot \int_I |K(x,y)| dy = \|g\|_\infty \cdot \int_I |K(x,y)| dy.$$

So,

$$\int_I |(\mathcal{L}g)(x)| w(x) dx \leq \|g\|_\infty \cdot \int_I \int_I |K(x,y)| w(x) dx dy,$$

which completes the proof of Proposition 4.1. □

Proposition 4.2. If $f \in L^\infty(I, w(x)dx)$ and $g \in L^\infty(I, w(x)dx)$, $w \geq 1$ a.e. on I, and

$$\int_I \int_I |K(x,y)| w(x) dx dy < \infty,$$

then the following Parseval-type relation holds

$$\int_I (\mathcal{F}f)(x) g(x) dx = \int_I f(x)(\mathcal{L}g)(x) dx. \tag{4.1}$$

Proof. In fact,

$$|(\mathcal{F}f)(x)| \leq \int_I |f(t)| \cdot |K(t,x)| dt$$
$$\leq ess\ sup_{t \in I} |f(t)| \cdot \int_I |K(t,x)| dt = \|f\|_\infty \cdot \int_I |K(t,x)| dt.$$

Therefore,

$$\int_I |(\mathcal{F}f)(x)||g(x)| dx \leq \|f\|_\infty \cdot \int_I \left(\int_I |K(t,x)| dt \right) |g(x)| dx$$
$$\leq \|f\|_\infty \cdot \|g\|_\infty \cdot \int_I \int_I |K(t,x)| dt dx$$
$$\leq M \cdot \|f\|_\infty \cdot \|g\|_\infty,$$

for some $M > 0$.

On the other hand,

$$\int_I |f(x)||(\mathcal{L}g)(x)| dx \leq M \cdot \|f\|_\infty \cdot \|g\|_\infty,$$

for some $M > 0$.

Now, using Fubini's theorem, we obtain (4.1). □

Therefore, for $f, g \in L^\infty(I, w(x)dx)$, we have

$$| < T_{\mathcal{F}f}, g > | \leq \int_I |(\mathcal{F}f)(x)| g(x)| dx$$
$$\leq M \cdot \|f\|_\infty \cdot \|g\|_\infty,$$

for some $M > 0$. So, $T_{\mathcal{F}f} \in (L^\infty(I, w(x)dx))'$.

On the other hand, for $f \in L^\infty(I, w(x)dx)$ and $g \in L^1(I, w(x)dx)$, we have

$$| < T_f, g > | \leq \int_I |f(x)||g(x)| dx$$
$$\leq \|f\|_\infty \cdot \|g\|_1,$$

from which we conclude that $T_f \in (L^1(I, w(x)dx))'$. Thus, we have proved the next result.

Corollary 4.1. If $f \in L^\infty(I, w(x)dx)$, $w \geq 1$ a.e. on I, and

$$\int_I \int_I |K(x,y)| w(x) dx dy < \infty,$$

then

$$\mathcal{L}' T_f = T_{\mathcal{F}f}$$

on $(L^\infty(I, w(x)dx))'$.

5 Particular cases : the index transforms and the operators with complex Gaussian kernels

5.1. The Kontorovich-Lebedev Transform

The Kontorovich-Lebedev transform [[9], Chap. 2] of a suitable function f is given by

$$(\mathcal{F}f)(y) = \int_0^\infty f(x) K_{iy}(x) dx, \quad y > 0,$$

where K_{iy} denotes the Bessel function of the third kind of purely imaginary order [[2], p. 4].

Here one takes the spaces $L^p(I, (1+x)^\gamma)$, $1 < p < \infty$, $\gamma \in \mathbb{R}$ and $I = (0, \infty)$. In this case, the results of Section 2 hold for $\gamma > p - 1$.

5.2. The Mehler-Fock Transform

The Mehler-Fock transform [[9], Chap. 3] of a suitable function f is given by

$$(\mathcal{F}f)(y) = \int_0^\infty f(x) P^{-n}_{-\frac{1}{2}+iy}(cosh\, x) dx, \quad y > 0, \quad n \in \mathbb{N} \cup \{0\},$$

where $P^{-n}_{-\frac{1}{2}+iy}$ is the Legendre function of first kind [[1], p. 122].

Here one takes the spaces $L^p(I, (1+x)^\gamma)$, $1 < p < \infty$, $\gamma \in \mathbb{R}$ and $I = (0, \infty)$. In this case, the results of Section 2 hold for $\gamma > p - 1$.

For the case $p = 1$ one takes the spaces $L^1(I, (1+x)^\gamma)$. Here, the results of Section 3 hold for $\gamma \geq 0$.

5.3. The ${}_2F_1-$Index Transform

The ${}_2F_1-$index transform was considered by Hayek, González, and Negrin in [4], [5], and [6]. This ${}_2F_1-$index transform of a suitable function f is given by

$$(\mathcal{F}f)(y) = \int_0^\infty f(x) \mathbf{F}(\mu, \alpha, y, x) dx, \quad y > 0,$$

where $\mathbf{F}(\mu, \alpha, y, x) = {}_2F_1(\mu + \frac{1}{2} + iy, \mu + \frac{1}{2} - iy; \mu + 1; -x) x^\alpha$, $\alpha, \mu \in \mathbb{C}$, $\Re\mu > -\frac{1}{2}$, and ${}_2F_1$ is the Gaussian hypergeometric function [[1], p. 59].

Here one takes the spaces $L^1(I, e^{\beta x} dx)$, $\beta \in \mathbb{R}$, and $I = (0, \infty)$. In this case, the results of Section 3 hold for $\beta \geq 0$, and $0 \leq \Re\alpha \leq \Re\mu + \frac{1}{2}$.

5.4. The operators with complex Gaussian kernels

The operator with complex Gaussian kernel (cf. [7], [8]) of a suitable function f is given by

$$(\mathcal{F}f)(y) = \int_{-\infty}^{+\infty} exp\{-\epsilon x^2 - \beta y^2 + 2\delta xy + \gamma x + \xi y\} f(x) dx,$$

where $y \in \mathbb{R}$, and $\epsilon, \beta, \delta, \gamma, \xi \in \mathbb{C}$.

Here one takes the spaces $L^p(\mathbb{R}, dx)$, $1 < p < \infty$. In this case,

$$\int_I \int_I |K(x,y)|^q dxdy = \int_{-\infty}^{+\infty} exp\{q\Re\xi\, y - q\Re\beta\, y^2\}$$

$$\cdot \int_{-\infty}^{+\infty} exp\{-q\Re\epsilon\, x^2 + q\Re\gamma\, x + 2q\Re\delta\, xy\} dxdy$$

$$= \int_{-\infty}^{+\infty} exp\{q\Re\xi\, y - q\Re\beta\, y^2\} \cdot \sqrt{\frac{\pi}{q \cdot \Re\epsilon}}$$

$$\cdot exp \left\{ \frac{1}{\Re\epsilon} \left(q(\Re\delta)^2\, y^2 + \frac{q(\Re\gamma)^2}{4} + q\Re\delta\, \Re\gamma\, y \right) \right\} dy,$$

whenever $\Re\epsilon > 0$. Now, for $\dfrac{(\Re\delta)^2}{\Re\epsilon} < \Re\beta$, this last integral is bounded. So, the results of Section 2 hold for $(\Re\delta)^2 < (\Re\epsilon) \cdot (\Re\beta)$ and $\Re\epsilon > 0$.

For the particular case when $p = 1$, one studies

$$ess\ sup_{x \in I}\ ess\ sup_{y \in I} |K(x,y)|$$

which is bounded for $(\Re\delta)^2 < \Re\beta$ and $\Re\epsilon > 1$ or, alternately, for $(\Re\delta)^2 < \Re\epsilon$ and $\Re\beta > 1$. Also for $\Re\delta = 0$, $\Re\epsilon > 0$ and $\Re\beta > 0$, the above expression is bounded.

Finally, for the particular case when $p = \infty$, one computes

$$\int_I \int_I |K(x,y)| dxdy$$

which is bounded for $(\Re\delta)^2 < (\Re\epsilon) \cdot (\Re\beta)$ and $\Re\epsilon > 0$.

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AN m-DIMENSIONAL VERSION OF DIFFERENCE OF TWO INTEGRAL MEANS FOR MAPPINGS OF THE HÖLDER TYPE

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ABSTRACT. Generalizations of estimations of difference of two multiple integral means for mappings of the Hölder type, are given. We establish four integral identities and use them to prove a number of inequalities for difference of two integral means of several variables.

1. INTRODUCTION

The following Ostrowski inequality is well known [7]:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\, L, \quad x \in [a,b], \tag{1.1}$$

where $f : [a,b] \to \mathbf{R}$ is a differentiable function such that $|f'(x)| \le L$, for every $x \in [a,b]$.

The Ostrowski inequality has been generalized over the last years in a number of ways. In 1984 J. Pečarić and B. Savić [9] generalized (1.1) for functions of several variables and they proved the following result.

Theorem 1. *Consider a real linear space X of real valued functions $f : Q \to \mathbf{R}$, where Q is a subset of $\mathbf{R}^m$, $m \in \mathbf{N}$, and assume that $\mathbf{1} \in X$ (here $\mathbf{1}$ denotes the constant function $\mathbf{x} \to \mathbf{1}$, $\mathbf{x} \in Q$). Let $A : X \to \mathbf{R}$ be a positive linear functional on X such that $A(\mathbf{1}) = \mathbf{1}$ and let $f, g \in X$ be such that $fg \in X$. Suppose that f satisfies the $\mathbf{r}$-Hölder condition ($\mathbf{r} = (r_1, \ldots, r_m)$)*

$$|f(\mathbf{x}) - f(\mathbf{y})| \le \sum_{i=1}^m L_i |x_i - y_i|^{r_i} \quad \text{for all } \mathbf{x}, \mathbf{y} \in Q, \tag{1.2}$$

where $L_i \ge 0$ and $0 < r_i \le 1$, $i = 1, \ldots, m$, are some constants. For any fixed $\mathbf{x} \in Q$, define $f_i : Q \to \mathbf{R}, \imath = 1, \ldots, m$, as

$$f_i(\mathbf{y}) = |y_i - x_i|^{r_i}, \ \mathbf{y} \in Q, \ i = 1, \ldots, m.$$

If $g(\mathbf{y}) \ge 0$ for all $\mathbf{y} \in Q$ and $A(g) > 0$, then

$$\left| f(\mathbf{x}) - \frac{A(fg)}{A(g)} \right| \le \sum_{i=1}^m L_i \frac{A(f_i g)}{A(g)}. \tag{1.3}$$

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A special case of Theorem 1 can be obtained in the following way, [9]. Take

$$Q = [\mathbf{a}, \mathbf{b}] := \prod_{i=1}^{m} [a_i, b_i], \ a_i, b_i \in \mathbf{R}, \ a_i < b_i, \ i = 1, \dots, m,$$

and let X be a linear space which contains all functions $f : [\mathbf{a}, \mathbf{b}] \to \mathbf{R}$ integrable on $[\mathbf{a}, \mathbf{b}]$. Define $A : X \to \mathbf{R}$ as

$$A(f) := \frac{1}{\prod_{i=1}^{m}(b_i - a_i)} \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{y}) d\mathbf{y}, \ \mathbf{y} \in X,$$

where $\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{y}) d\mathbf{y} = \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} f(y_1, \dots, y_m) dy_m \dots dy_1$.

Then (1.3) becomes

$$\left| f(\mathbf{x}) - \frac{\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{y}) g(\mathbf{y}) d\mathbf{y}}{\int_{\mathbf{a}}^{\mathbf{b}} g(\mathbf{y}) d\mathbf{y}} \right| \le \sum_{i=1}^{m} L_i \frac{\int_{\mathbf{a}}^{\mathbf{b}} |y_i - x_i|^{r_i} g(\mathbf{y}) d\mathbf{y}}{\int_{\mathbf{a}}^{\mathbf{b}} g(\mathbf{y}) d\mathbf{y}}, \ \mathbf{x} \in [\mathbf{a}, \mathbf{b}]. \tag{1.4}$$

If f satisfies the condition (1.2) with $r_1 = \cdots = r_m = 1$, then (1.4) can be rewritten as

$$\left| f(\mathbf{x}) - \frac{\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{y}) g(\mathbf{y}) d\mathbf{y}}{\int_{\mathbf{a}}^{\mathbf{b}} g(\mathbf{y}) d\mathbf{y}} \right| \le \sum_{i=1}^{m} L_i \frac{\int_{\mathbf{a}}^{\mathbf{b}} |y_i - x_i| g(\mathbf{y}) d\mathbf{y}}{\int_{\mathbf{a}}^{\mathbf{b}} g(\mathbf{y}) d\mathbf{y}}, \tag{1.5}$$

When $g(\mathbf{y}) = 1$ for all $\mathbf{y} \in [\mathbf{a}, \mathbf{b}]$, then (1.5) reduces to

$$\left| f(\mathbf{x}) - \frac{\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{y}) d\mathbf{y}}{\prod_{i=1}^{m}(b_i - a_i)} \right| \le \sum_{i=1}^{m} \left[\frac{1}{4} + \frac{(x_i - ((a_i + b_i)/2))^2}{(b_i - a_i)^2} \right] (b_i - a_i) L_i$$

which is for $m = 1$ Ostrowski inequality (1.1).

Also, it should be noted that the inequality (1.4), along with the special case when $g(\mathbf{y}) = 1$ for all $\mathbf{y} \in [\mathbf{a}, \mathbf{b}]$, was rediscovered in the paper [4].

In this paper we will generalize the results from [3] and [5], where is estimated the difference of the two integral means for absolutely continuous mappings whose first derivative is in $L_\infty[a, b]$. See also [8] and [1]. M. Matić and J. Pečarić in [5] proved the following result which is in spirit of our results, so we use it as a initial result:

Theorem 2. *Let $a, b, c, d \in \mathbf{R}$, be such that*

$$a \le c < d \le b, \ c - a + b - d > 0.$$

(i) If $f : [a, b] \to \mathbf{R}$ is L-Lipschitzian on $[a, b]$, with some constant $L > 0$, then

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(s) ds \right| \le \frac{(c-a)^2 + (b-d)^2}{2(c - a + b - d)} L. \tag{1.6}$$

(ii) If $f_0 : [a, b] \to \mathbf{R}$ is defined as

$$f_0(t) = |t - s_0|, \ t \in [a, b],$$

where

$$s_0 = \frac{bc - ad}{c - a + b - d},$$

then f_0 is 1-Lipschitzian on $[a, b]$ and we have

$$\left| \frac{1}{b-a} \int_a^b f_0(t) dt - \frac{1}{d-c} \int_c^d f_0(s) ds \right| = \frac{(c-a)^2 + (b-d)^2}{2(c - a + b - d)}.$$

Note that for $c = d = x$ we can assume $\frac{1}{d-c}\int_c^d f(s)ds = f(x)$, as a limit case, so that (1.6) reduces to the Ostrowski inequality (1.1). So, inequality (1.6) can be regarded as a natural generalization of Ostrowski inequality (1.1).

We will give generalizations of estimations of difference of two multiple integral means for mappings of the Hölder type. Also, we will establish four integral identities and use them to prove a number of inequalities for difference of two integral means of several variables for functions of class $C^n([\mathbf{a},\mathbf{b}])$ and $C^{n+1}([\mathbf{a},\mathbf{b}])$. We will make it in two cases. First is when $\mathbf{x} \in [\mathbf{c},\mathbf{d}] \subseteq [\mathbf{a},\mathbf{b}]$ and the other when $\mathbf{x} \in [\mathbf{c},\mathbf{b}] = [\mathbf{a},\mathbf{b}] \cap [\mathbf{c},\mathbf{d}]$.

2. A generalizations for mappings of the Hölder type

We start with the following inequalities for estimations of diference of two integral means for mapping which satisfies the $\mathbf{r}$-Hölder type condition (1.2).

2.1. Case $[\mathbf{c},\mathbf{d}] \subseteq [\mathbf{a},\mathbf{b}]$.

Theorem 3. *Let $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \mathbf{R}^m$, are such that $[\mathbf{c},\mathbf{d}] \subseteq [\mathbf{a},\mathbf{b}]$, i.e. $a_i \le c_i < d_i \le b_i$, $c_i - a_i + b_i - d_i > 0$ for all $i = 1,\dots,m$. Assume that the mapping $f : [\mathbf{a},\mathbf{b}] \to \mathbf{R}$ satisfies condition (1.2). Then we have inequality*

$$\begin{aligned}
&\left|\frac{1}{\prod_{i=1}^m (b_i - a_i)}\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{t})d\mathbf{t} - \frac{1}{\prod_{i=1}^m (d_i - c_i)}\int_{\mathbf{c}}^{\mathbf{d}} f(\mathbf{s})d\mathbf{s}\right| \\
\le\ & \sum_{i=1}^m \frac{L_i}{r_i+1}\,\frac{(c_i-a_i)^{r_i+1} + (b_i-d_i)^{r_i+1}}{c_i - a_i + b_i - d_i}. \qquad (2.1)
\end{aligned}$$

Proof. By the substitution

$$t_i = \frac{b_i - a_i}{d_i - c_i}s_i - \frac{b_ic_i - a_id_i}{d_i - c_i},\ \ s_i \in [c_i, d_i]$$

we get

$$\begin{aligned}
&\frac{1}{\prod_{i=1}^m (b_i - a_i)}\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{t})d\mathbf{t} = \frac{1}{\prod_{i=1}^m (d_i - c_i)} \\
\cdot\ & \int_{\mathbf{c}}^{\mathbf{d}} f\left(\frac{b_1-a_1}{d_1-c_1}s_1 - \frac{b_1c_1-a_1d_1}{d_1-c_1},\dots,\frac{b_m-a_m}{d_m-c_m}s_m - \frac{b_mc_m-a_md_m}{d_m-c_m}\right)d\mathbf{s}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\frac{1}{\prod_{i=1}^m (b_i - a_i)}\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{t})d\mathbf{t} - \frac{1}{\prod_{i=1}^m (d_i - c_i)}\int_{\mathbf{c}}^{\mathbf{d}} f(\mathbf{s})d\mathbf{s}\right| = \frac{1}{\prod_{i=1}^m (d_i - c_i)} \\
\cdot\ & \left|\int_{\mathbf{c}}^{\mathbf{d}} \left[f\left(\frac{b_1-a_1}{d_1-c_1}s_1 - \frac{b_1c_1-a_1d_1}{d_1-c_1},\dots,\frac{b_m-a_m}{d_m-c_m}s_m - \frac{b_mc_m-a_md_m}{d_m-c_m}\right) - f(\mathbf{s})\right]d\mathbf{s}\right| \\
\le\ & \frac{1}{\prod_{i=1}^m (d_i - c_i)} \\
\cdot\ & \int_{\mathbf{c}}^{\mathbf{d}} \left|f\left(\frac{b_1-a_1}{d_1-c_1}s_1 - \frac{b_1c_1-a_1d_1}{d_1-c_1},\dots,\frac{b_m-a_m}{d_m-c_m}s_m - \frac{b_mc_m-a_md_m}{d_m-c_m}\right) - f(\mathbf{s})\right|d\mathbf{s}
\end{aligned}$$

$$\leq \frac{1}{\prod_{i=1}^m (d_i - c_i)} \sum_{i=1}^m L_i \int_{\mathbf{c}}^{\mathbf{d}} \left| \frac{c_i - a_i + b_i - d_i}{d_i - c_i} s_i - \frac{b_i c_i - a_i d_i}{d_i - c_i} \right|^{r_i} d\mathbf{s}$$

$$= \sum_{i=1}^m \frac{L_i}{d_i - c_i} \frac{(c_i - a_i + b_i - d_i)^{r_i}}{(d_i - c_i)^{r_i}} \int_{c_i}^{d_i} |s_i - s_0^i|^{r_i} ds_i, \tag{2.2}$$

as

$$\int_{c_1}^{d_1} \cdots \int_{c_n}^{d_n} h(s_i) ds_n \dots ds_1 = \prod_{\substack{j=1 \\ j \neq i}}^m (d_j - c_j) \int_{c_i}^{d_i} h(s_i) ds_i$$

for some function h.

Further, $s_0^i = (b_i c_i - a_i d_i)/(c_i - a_i + b_i - d_i)$ and we have

$$s_0^i - c_i = \frac{d_i - c_i}{c_i - a_i + b_i - d_i}(c_i - a_i) \geq 0 \quad \text{and} \quad d_i - s_0^i = \frac{d_i - c_i}{c_i - a_i + b_i - d_i}(b_i - d_i) \geq 0,$$

which implies that $s_0^i \in [c_i, d_i]$ and

$$\begin{aligned} \int_{c_i}^{d_i} |s_i - s_0^i|^{r_i} ds_i &= \int_{c_i}^{s_0^i} (s_0^i - s_i)^{r_i} ds_i + \int_{s_0^i}^{d_i} (s_i - s_0^i)^{r_i} ds_i \\ &= \frac{1}{r_i + 1} [(s_0^i - c_i)^{r_i + 1} + (d_i - s_0^i)^{r_i + 1}] \\ &= \frac{(d_i - c_i)^{r_i + 1}}{(r_i + 1)(c_i - a_i + b_i - d_i)^{r_i + 1}} [(c_i - a_i)^{r_i + 1} + (b_i - d_i)^{r_i + 1}]. \end{aligned}$$

Substituting this in (2.2) we get (2.1). □

An important particular case is one for which the mapping f is Lipschitzian, i.e. we have $r_1 = \cdots = r_m = 1$ in condition (1.2). Thus, we have following corollary.

Corollary 1. *Let f be a Lipschitzian mapping with the constants L_i. Then we have*

$$\left| \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{t}) d\mathbf{t} - \frac{1}{\prod_{i=1}^m (d_i - c_i)} \int_{\mathbf{c}}^{\mathbf{d}} f(\mathbf{s}) d\mathbf{s} \right| \leq \sum_{i=1}^m \frac{L_i}{2} \frac{(c_i - a_i)^2 + (b_i - d_i)^2}{c_i - a_i + b_i - d_i}. \tag{2.3}$$

The constant $1/2$ *is the best possible.*

Proof. Put $r_i = 1$ $(i = 1, \dots, m)$ in (2.1) to get inequality (2.3).

To prove the best possibility of the constant $1/2$, assume that the inequality (2.3) holds for some positive constant $c > 0$, i.e.

$$\left| \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{t}) d\mathbf{t} - \frac{1}{\prod_{i=1}^m (d_i - c_i)} \int_{\mathbf{c}}^{\mathbf{d}} f(\mathbf{s}) d\mathbf{s} \right| \leq \sum_{i=1}^m c \cdot L_i \frac{(c_i - a_i)^2 + (b_i - d_i)^2}{c_i - a_i + b_i - d_i}. \tag{2.4}$$

Choose $f(t_1, \dots, t_m) = t_i$ $(i = 1, \dots, m)$. Then, by (2.4), we get

$$\frac{a_i + b_i}{2} - \frac{c_i + d_i}{2} \leq c \frac{(c_i - a_i)^2 + (b_i - d_i)^2}{c_i - a_i + b_i - d_i}.$$

Put $c_i = a_i$ to get $c \geq 1/2$ and the best possibility is proved. □

Remark 1. *If in Theorem 3 and Corollary 1 we put $\mathbf{c} = \mathbf{d} = \mathbf{x}$ as a limit case we get generalized Ostrowski inequalities (see* [4]*).*

2.2. **Case $[\mathbf{a},\mathbf{b}]\cap[\mathbf{c},\mathbf{d}]=[\mathbf{c},\mathbf{b}]$.**

Theorem 4. *Let $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\mathbf{R}^m$, are such that $[\mathbf{a},\mathbf{b}]\cap[\mathbf{c},\mathbf{d}]=[\mathbf{c},\mathbf{b}]$, i.e. $a_i\le c_i<b_i\le d_i$ for all $i=1,\dots,m$. Assume that the mapping $f:[\mathbf{a},\mathbf{d}]\to\mathbf{R}$ satisfies condition (1.2). Then we have inequality*

$$\begin{aligned}&\left|\frac{1}{\prod_{i=1}^m(b_i-a_i)}\int_{\mathbf{a}}^{\mathbf{b}}f(\mathbf{t})d\mathbf{t}-\frac{1}{\prod_{i=1}^m(d_i-c_i)}\int_{\mathbf{c}}^{\mathbf{d}}f(\mathbf{s})d\mathbf{s}\right|\\ \le\ &\sum_{i=1}^m\frac{L_i}{r_i+1}\frac{(c_i-a_i)^{r_i+1}-(d_i-b_i)^{r_i+1}}{c_i-a_i+b_i-d_i}.\end{aligned}\tag{2.5}$$

Proof. By the substitution

$$t_i=\frac{b_i-a_i}{b_i-c_i}p_i-\frac{b_ic_i-a_ib_i}{b_i-c_i},\ s_i=\frac{d_i-c_i}{b_i-c_i}p_i-\frac{d_ic_i-b_ic_i}{b_i-c_i},\ p_i\in[b_i,c_i]$$

we get

$$\begin{aligned}&\frac{1}{\prod_{i=1}^m(b_i-a_i)}\int_{\mathbf{a}}^{\mathbf{b}}f(\mathbf{t})d\mathbf{t}=\frac{1}{\prod_{i=1}^m(b_i-c_i)}\\ \cdot\ &\int_{\mathbf{c}}^{\mathbf{b}}f\left(\frac{b_1-a_1}{b_1-c_1}p_1-\frac{b_1c_1-a_1b_1}{b_1-c_1},\dots,\frac{b_m-a_m}{b_m-c_m}p_m-\frac{b_mc_m-a_mb_m}{b_m-c_m}\right)d\mathbf{p}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\prod_{i=1}^m(d_i-c_i)}\int_{\mathbf{c}}^{\mathbf{d}}f(\mathbf{s})d\mathbf{s}=\frac{1}{\prod_{i=1}^m(b_i-c_i)}\\ \cdot\ &\int_{\mathbf{c}}^{\mathbf{b}}f\left(\frac{d_1-c_1}{b_1-c_1}p_1-\frac{d_1c_1-b_1c_1}{b_1-c_1},\dots,\frac{d_m-c_m}{b_m-c_m}p_m-\frac{d_mc_m-b_mc_m}{b_m-c_m}\right)d\mathbf{p}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left|\frac{1}{\prod_{i=1}^m(b_i-a_i)}\int_{\mathbf{a}}^{\mathbf{b}}f(\mathbf{t})d\mathbf{t}-\frac{1}{\prod_{i=1}^m(d_i-c_i)}\int_{\mathbf{c}}^{\mathbf{d}}f(\mathbf{s})d\mathbf{s}\right|=\frac{1}{\prod_{i=1}^m(b_i-c_i)}\\ \cdot\ &\left|\int_{\mathbf{c}}^{\mathbf{b}}\left[f\left(\frac{b_1-a_1}{b_1-c_1}p_1-\frac{b_1c_1-a_1b_1}{b_1-c_1},\dots,\frac{b_m-a_m}{b_m-c_m}p_m-\frac{b_mc_m-a_mb_m}{b_m-c_m}\right)\right.\right.\\ -\ &\left.\left.f\left(\frac{d_1-c_1}{b_1-c_1}p_1-\frac{d_1c_1-b_1c_1}{b_1-c_1},\dots,\frac{d_m-c_m}{b_m-c_m}p_m-\frac{d_mc_m-b_mc_m}{b_m-c_m}\right)\right]d\mathbf{p}\right|\\ \le\ &\frac{1}{\prod_{i=1}^m(b_i-c_i)}\\ \cdot\ &\int_{\mathbf{c}}^{\mathbf{b}}\left|f\left(\frac{b_1-a_1}{b_1-c_1}p_1-\frac{b_1c_1-a_1b_1}{b_1-c_1},\dots,\frac{b_m-a_m}{b_m-c_m}p_m-\frac{b_mc_m-a_mb_m}{b_m-c_m}\right)\right.\\ -\ &\left.f\left(\frac{d_1-c_1}{b_1-c_1}p_1-\frac{d_1c_1-b_1c_1}{b_1-c_1},\dots,\frac{d_m-c_m}{b_m-c_m}p_m-\frac{d_mc_m-b_mc_m}{b_m-c_m}\right)\right|d\mathbf{p}\\ \le\ &\frac{1}{\prod_{i=1}^m(b_i-c_i)}\sum_{i=1}^mL_i\int_{\mathbf{c}}^{\mathbf{b}}\left|\frac{c_i-a_i+b_i-d_i}{b_i-c_i}p_i-\frac{b_ic_i-a_ib_i-d_ic_i+b_ic_i}{b_i-c_i}\right|^{r_i}d\mathbf{p}\\ =\ &\sum_{i=1}^m\frac{L_i}{b_i-c_i}\frac{|c_i-a_i+b_i-d_i|^{r_i}}{(b_i-c_i)^{r_i}}\int_{c_i}^{b_i}|p_i-p_0^i|^{r_i}dp_i.\end{aligned}\tag{2.6}$$

With no loss in generality we can take that $c_i - a_i + b_i - d_i \geq 0$. Further, $p_0^i = (b_ic_i - a_ib_i - d_ic_i + b_ic_i)/(c_i - a_i + b_i - d_i)$ and we have

$$p_0^i - c_i = \frac{b_i - c_i}{c_i - a_i + b_i - d_i}(c_i - a_i) \geq 0 \quad \text{and} \quad p_0^i - b_i = \frac{b_i - c_i}{c_i - a_i + b_i - d_i}(d_i - b_i) \geq 0,$$

which implies that $p_0^i \notin [c_i, b_i]$ and

$$\begin{aligned} \int_{c_i}^{b_i} |p_i - p_0^i|^{r_i} dp_i &= \int_{c_i}^{b_i} (p_0^i - p_i)^{r_i} dp_i = \frac{1}{r_i + 1}[(p_0^i - c_i)^{r_i+1} - (p_0^i - b_i)^{r_i+1}] \\ &= \frac{(b_i - c_i)^{r_i+1}}{(r_i + 1)(c_i - a_i + b_i - d_i)^{r_i+1}}[(c_i - a_i)^{r_i+1} - (d_i - b_i)^{r_i+1}]. \end{aligned}$$

Substituting this in (2.6) we get (2.5). □

An important particular case is one for which the mapping f is Lipschitzian, i.e. we have $r_1 = \cdots = r_m = 1$ in condition (1.2). Thus, we have following corollary.

Corollary 2. *Let f be a Lipschitzian mapping with the constants L_i. Then we have*

$$\left| \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{t}) d\mathbf{t} - \frac{1}{\prod_{i=1}^m (d_i - c_i)} \int_{\mathbf{c}}^{\mathbf{d}} f(\mathbf{s}) d\mathbf{s} \right| \leq \sum_{i=1}^m L_i \frac{c_i - a_i - b_i + d_i}{2}. \tag{2.7}$$

The constant $1/2$ is the best possible.

Proof. Put $r_i = 1$ $(i = 1, \ldots, m)$ in (2.5) to get inequality (2.7).

To prove the best possibility of the constant $1/2$, assume that the inequality (2.7) holds for some positive constant $c > 0$, i.e.

$$\left| \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{t}) d\mathbf{t} - \frac{1}{\prod_{i=1}^m (d_i - c_i)} \int_{\mathbf{c}}^{\mathbf{d}} f(\mathbf{s}) d\mathbf{s} \right| \leq \sum_{i=1}^m c \cdot L_i (c_i - a_i - b_i + d_i). \tag{2.8}$$

Choose $f(t_1, \ldots, t_m) = t_i$ $(i = 1, \ldots, m)$. Then, by (2.8), we get

$$\left| \frac{a_i + b_i}{2} - \frac{c_i + d_i}{2} \right| \leq c(c_i - a_i - b_i + d_i),$$

which is $c \geq 1/2$ and the best possibility is proved. □

Remark 2. *If in Theorem 4 and Corollary 2 we put $\mathbf{c} = \mathbf{b} = \mathbf{x}$ we get inequalities*

$$\begin{aligned} &\left| \frac{1}{\prod_{i=1}^m (x_i - a_i)} \int_{\mathbf{a}}^{\mathbf{x}} f(\mathbf{t}) d\mathbf{t} - \frac{1}{\prod_{i=1}^m (d_i - x_i)} \int_{\mathbf{x}}^{\mathbf{d}} f(\mathbf{s}) d\mathbf{s} \right| \\ \leq &\sum_{i=1}^m \frac{L_i}{r_i + 1} \frac{(x_i - a_i)^{r_i+1} - (d_i - x_i)^{r_i+1}}{2x_i - a_i - d_i} \end{aligned} \tag{2.9}$$

and

$$\left| \frac{1}{\prod_{i=1}^m (x_i - a_i)} \int_{\mathbf{a}}^{\mathbf{x}} f(\mathbf{t}) d\mathbf{t} - \frac{1}{\prod_{i=1}^m (d_i - x_i)} \int_{\mathbf{x}}^{\mathbf{d}} f(\mathbf{s}) d\mathbf{s} \right| \leq \sum_{i=1}^m L_i \frac{d_i - a_i}{2}. \tag{2.10}$$

For $m = 1$ in the above inequalities we get results from [1].

Remark 3. *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbf{R}^m$ *are such that for* $i \in S \subseteq \{1, \dots, m\}$, $[c_i, d_i] \subseteq [a_i, b_i]$ *and for* $i \in \{1, \dots, m\} \setminus S$, $[a_i, b_i] \cap [c_i, d_i] = [c_i, b_i]$, *then from Theorems 3 and 4 we get inequality*

$$\left| \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{t}) d\mathbf{t} - \frac{1}{\prod_{i=1}^m (d_i - c_i)} \int_{\mathbf{c}}^{\mathbf{d}} f(\mathbf{s}) d\mathbf{s} \right| \leq \sum_{i=1}^m \frac{L_i}{r_i + 1} \frac{F(a_i, b_i, c_i, d_i)}{c_i - a_i + b_i - d_i},$$

where

$$F(a_i, b_i, c_i, d_i) = (c_i - a_i)^{r_i+1} + (b_i - d_i)^{r_i+1} \quad \textit{if} \quad [c_i, d_i] \subseteq [a_i, b_i]$$

and

$$F(a_i, b_i, c_i, d_i) = (c_i - a_i)^{r_i+1} - (d_i - b_i)^{r_i+1} \quad \textit{if} \quad [a_i, b_i] \cap [c_i, d_i] = [c_i, b_i].$$

3. Four integral identities

M. Matić, J. Pečarić and N. Ujević [6] considered generalizations of Theorem 1 for functions of class $C^n(Q)$ and $C^{n+1}(Q)$ (see also [2]). They proved two integral identities for such functions.

Let Q be any compact and convex subset of $\mathbf{R}^m$, $m \in \mathbf{N}$. A weight function on Q is any function $w : Q \to [0, \infty)$ which is integrable on Q and

$$\int_Q w(\mathbf{y}) d\mathbf{y} > 0.$$

For given m-tuple $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$, $\alpha_i \in \mathbf{N} \cup \{0\}$, $i = 1, \dots, m$, we use the notation

$$|\boldsymbol{\alpha}| := \sum_{i=1}^m \alpha_i \quad \text{and } \boldsymbol{\alpha}! := \prod_{i=1}^m \alpha_i! = \alpha_1! \cdots \alpha_m!.$$

Also, for any $\mathbf{z} = (z_1, \dots, z_m) \in \mathbf{R}^m$, we set

$$\mathbf{z}^{\boldsymbol{\alpha}} := \prod_{i=1}^m z_i^{\alpha_i} = z_1^{\alpha_1} \cdots z_m^{\alpha_m}.$$

Here we assume the convention $0^0 = 1$. With such notation the following multinomial formula is valid:

$$\left(\sum_{i=1}^m z_i \right)^n = \sum_{|\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!} \mathbf{z}^{\boldsymbol{\alpha}}, \; n \in \mathbf{N}.$$

Also, for given m-tuple $\mathbf{r} = (r_1, \dots, r_m)$, $r_i \in [0, \infty)$, $i = 1, \dots, m$, we set

$$|\mathbf{z}| := (|z_1|, \dots, |z_m|) \quad \text{and} \quad |\mathbf{z}|^{\mathbf{r}} := \prod_{i=1}^m |z_i|^{r_i} = |z_1|^{r_1} \cdots |z_m|^{r_m},$$

again with convention $0^0 = 1$.

If a weight function $w : Q \to [0, \infty)$ is given, for any fixed $\mathbf{x} \in Q$ we define the $\mathbf{x}$-centred moment $E(\mathbf{x}, Q; w)$ of order $\boldsymbol{\alpha}$, of the set Q with respect to w as

$$E_{\boldsymbol{\alpha}}(\mathbf{x}, Q; w) := \int_Q (\mathbf{y} - \mathbf{x})^{\boldsymbol{\alpha}} w(\mathbf{y}) d\mathbf{y}.$$

Note that $E_{\mathbf{0}}(\mathbf{x},Q;w)=\int_Q w(\mathbf{y})d\mathbf{y}$, where $\mathbf{0}=(0,\dots,0)$. In the special case when $w(\mathbf{y})=\mathbf{1}$ for all $\mathbf{y}\in Q=[\mathbf{a},\mathbf{b}]$, where $\mathbf{a}=(a_1,\dots,a_m)$ and $\mathbf{b}=(b_1,\dots,b_m)$ are such that $a_i<b_i,\ i=1,\dots,m$ and

$$[\mathbf{a},\mathbf{b}]:=\prod_{i=1}^m[a_i,b_i]=\{(x_1,\dots,x_m):a_i\le x_i\le b_i,\ i=1,\dots,m\},$$

an easy calculation gives

$$E_{\boldsymbol{\alpha}}(\mathbf{x},[\mathbf{a},\mathbf{b}])=\prod_{i=1}^m\int_{a_i}^{b_i}(y_i-x_i)^{\alpha_i}dy_i=\prod_{i=1}^m\frac{(b_i-x_i)^{\alpha_i+1}+(-1)^{\alpha_i}(x_i-a_i)^{\alpha_i+1}}{\alpha_i+1}.$$

Next suppose that $f:V\to\mathbf{R}$ is any function defined on an open set $V\subset\mathbf{R}^m$ which contains Q as a subset. If for some $k\in\mathbf{N}$ partial derivatives $f_{\boldsymbol{\alpha}}(\mathbf{y})$ exist for all $\mathbf{y}\in Q$ and for all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}|\le k$, then we can define

$$\mathcal{R}_k(\mathbf{x},f;w):=\int_Q f(\mathbf{y})w(\mathbf{y})d\mathbf{y}-f(\mathbf{x})\int_Q w(\mathbf{y})d\mathbf{y}-\sum_{j=1}^k\sum_{|\boldsymbol{\alpha}|=j}\frac{f_{\boldsymbol{\alpha}}(\mathbf{x})}{\boldsymbol{\alpha}!}E_{\boldsymbol{\alpha}}(\mathbf{x},Q;w),$$

where $\mathbf{x}\in Q$ is any fixed element. Also, we set

$$\mathcal{R}_0(\mathbf{x},f;w):=\int_Q f(\mathbf{y})w(\mathbf{y})d\mathbf{y}-f(\mathbf{x})\int_Q w(\mathbf{y})d\mathbf{y},\ \mathbf{x}\in Q.$$

Theorem 5. *Let $f:V\to\mathbf{R}$ be a function defined on an open subset V of $\mathbf{R}^m$, $m\in\mathbf{N}$. Let Q be any compact and convex subset of V, and let $w:Q\to[0,\infty)$ be a weight function on Q.*
(i) If $f\in C^n(Q)$ for some $n\in\mathbf{N}$, then for any $\mathbf{x}\in Q$ we have

$$\begin{aligned}\mathcal{R}_n(\mathbf{x},f;w) &= \sum_{|\boldsymbol{\alpha}|=n}\frac{n}{\boldsymbol{\alpha}!}\int_Q(\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} \\ &\quad\cdot\left\{\int_0^1[f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))-f_{\boldsymbol{\alpha}}(\mathbf{x})](1-t)^{n-1}dt\right\}w(\mathbf{y})d\mathbf{y}.\end{aligned}\tag{3.1}$$

(ii) If $f\in C^{n+1}(Q)$ for some $n\in\mathbf{N}\cup\{0\}$, then for any $\mathbf{x}\in Q$ we have

$$\begin{aligned}\mathcal{R}_n(\mathbf{x},f;w) &= \sum_{|\boldsymbol{\alpha}|=n+1}\frac{n+1}{\boldsymbol{\alpha}!}\int_Q(\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} \\ &\quad\cdot\left\{\int_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(1-t)^n dt\right\}w(\mathbf{y})d\mathbf{y}.\end{aligned}\tag{3.2}$$

Now, using identities (3.1) and (3.2) we will establish two new integral identities.

Theorem 6. *Let $f:V\to\mathbf{R}$ be a function defined on an open subset V of $\mathbf{R}^m$, $m\in\mathbf{N}$. Let Q_1 and Q_2 be compact and convex subsets of V, $Q_1\cap Q_2\neq\emptyset$ and let $w:Q_1\to[0,\infty)$ be a weight function on Q_1 and $u:Q_2\to[0,\infty)$ be a weight function on Q_2.*

(i) If $f \in C^n(Q_1 \cup Q_2)$ for some $n \in \mathbf{N}$, then for any $\mathbf{x} \in Q_1 \cap Q_2$ we have

$$\begin{aligned}
& \frac{\int_{Q_1} f(\mathbf{y})w(\mathbf{y})d\mathbf{y}}{\int_{Q_1} w(\mathbf{y})d\mathbf{y}} - \frac{\int_{Q_2} f(\mathbf{y})u(\mathbf{y})d\mathbf{y}}{\int_{Q_2} u(\mathbf{y})d\mathbf{y}} \\
- & \sum_{j=1}^{k} \sum_{|\boldsymbol{\alpha}|=j} \frac{f_{\boldsymbol{\alpha}}(\mathbf{x})}{\boldsymbol{\alpha}!} \left[\frac{E_{\boldsymbol{\alpha}}(\mathbf{x}, Q_1; w)}{\int_{Q_1} w(\mathbf{y})d\mathbf{y}} - \frac{E_{\boldsymbol{\alpha}}(\mathbf{x}, Q_2; u)}{\int_{Q_2} u(\mathbf{y})d\mathbf{y}} \right] \\
= & \sum_{|\boldsymbol{\alpha}|=n} \frac{n}{\boldsymbol{\alpha}!} \left[\frac{\int_{Q_1} (\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} \cdot \left\{ \int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})](1-t)^{n-1} dt \right\} w(\mathbf{y})d\mathbf{y}}{\int_{Q_1} w(\mathbf{y})d\mathbf{y}} \right. \\
- & \left. \frac{\int_{Q_2} (\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} \cdot \left\{ \int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})](1-t)^{n-1} dt \right\} u(\mathbf{y})d\mathbf{y}}{\int_{Q_2} u(\mathbf{y})d\mathbf{y}} \right]. \qquad (3.3)
\end{aligned}$$

(ii) If $f \in C^{n+1}(Q_1 \cup Q_2)$ for some $n \in \mathbf{N} \cup \{0\}$, then for any $\mathbf{x} \in Q_1 \cap Q_2$ we have

$$\begin{aligned}
& \frac{\int_{Q_1} f(\mathbf{y})w(\mathbf{y})d\mathbf{y}}{\int_{Q_1} w(\mathbf{y})d\mathbf{y}} - \frac{\int_{Q_2} f(\mathbf{y})u(\mathbf{y})d\mathbf{y}}{\int_{Q_2} u(\mathbf{y})d\mathbf{y}} \\
- & \sum_{j=1}^{k} \sum_{|\boldsymbol{\alpha}|=j} \frac{f_{\boldsymbol{\alpha}}(\mathbf{x})}{\boldsymbol{\alpha}!} \left[\frac{E_{\boldsymbol{\alpha}}(\mathbf{x}, Q_1; w)}{\int_{Q_1} w(\mathbf{y})d\mathbf{y}} - \frac{E_{\boldsymbol{\alpha}}(\mathbf{x}, Q_2; u)}{\int_{Q_2} u(\mathbf{y})d\mathbf{y}} \right] \\
= & \sum_{|\boldsymbol{\alpha}|=n} \frac{n+1}{\boldsymbol{\alpha}!} \left[\frac{\int_{Q_1} (\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} \cdot \left\{ \int_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(1-t)^{n} dt \right\} w(\mathbf{y})d\mathbf{y}}{\int_{Q_1} w(\mathbf{y})d\mathbf{y}} \right. \\
- & \left. \frac{\int_{Q_2} (\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} \cdot \left\{ \int_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(1-t)^{n} dt \right\} u(\mathbf{y})d\mathbf{y}}{\int_{Q_2} u(\mathbf{y})d\mathbf{y}} \right]. \qquad (3.4)
\end{aligned}$$

Proof. First we write the identities (3.1) and (3.2) for Q_1 and for Q_2. Then we divide them with $\int_{Q_1} w(\mathbf{y})d\mathbf{y}$ and $\int_{Q_1} u(\mathbf{y})d\mathbf{y}$ respectively, subtract them and get the above statements. □

In the special case when $w(\mathbf{y}) = u(\mathbf{y}) = 1,\ Q_1 = [\mathbf{a}, \mathbf{b}]$ and $Q_2 = [\mathbf{c}, \mathbf{d}]$ we get four identities. The first two are when $\mathbf{x} \in [\mathbf{c}, \mathbf{d}] \subseteq [\mathbf{a}, \mathbf{b}]$ and the second are when $\mathbf{x} \in [\mathbf{c}, \mathbf{b}] = [\mathbf{a}, \mathbf{b}] \cap [\mathbf{c}, \mathbf{d}]$.

Corollary 3. *(i) Let $f : [\mathbf{a}, \mathbf{b}] \to \mathbf{R}$ be such that $f \in C^n([\mathbf{a}, \mathbf{b}])$ for some $n \in \mathbf{N}$. If $[\mathbf{c}, \mathbf{d}] \subseteq [\mathbf{a}, \mathbf{b}]$, then for every $\mathbf{x} \in [\mathbf{c}, \mathbf{d}]$ we have*

$$\begin{aligned}
\mathcal{O}(\mathbf{x}, f) \ = \ & \sum_{|\boldsymbol{\alpha}|=n} \frac{n}{\boldsymbol{\alpha}!} \int_{\mathbf{a}}^{\mathbf{b}} K_m(\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} \\
\cdot & \left\{ \int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})](1-t)^{n-1} dt \right\} d\mathbf{y}, \qquad (3.5)
\end{aligned}$$

where

$$\begin{aligned}\mathcal{O}(\mathbf{x},f) &= \frac{1}{\prod_{i=1}^m (b_i-a_i)}\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{y})d\mathbf{y} - \frac{1}{\prod_{i=1}^m (d_i-c_i)}\int_{\mathbf{c}}^{\mathbf{d}} f(\mathbf{y})d\mathbf{y} \\ &- \sum_{j=1}^{n}\sum_{|\boldsymbol{\alpha}|=j}\frac{f_{\boldsymbol{\alpha}}(\mathbf{x})}{\boldsymbol{\alpha}!}\left[\prod_{i=1}^m \frac{(b_i-x_i)^{\alpha_i+1}+(-1)^{\alpha_i}(x_i-a_i)^{\alpha_i+1}}{(b_i-a_i)(\alpha_i+1)}\right. \\ &- \left.\prod_{i=1}^m \frac{(d_i-x_i)^{\alpha_i+1}+(-1)^{\alpha_i}(x_i-c_i)^{\alpha_i+1}}{(d_i-c_i)(\alpha_i+1)}\right]\end{aligned}$$

and

$$K_m = \begin{cases} \frac{1}{\prod_{i=1}^m (b_i-a_i)} - \frac{1}{\prod_{i=1}^m (d_i-c_i)}, & \text{if } \mathbf{y}\in(\mathbf{c},\mathbf{d}], \\ \frac{1}{\prod_{i=1}^m (b_i-a_i)}, & \text{else.}\end{cases} \tag{3.6}$$

(ii) If $f \in C^{n+1}([\mathbf{a},\mathbf{b}])$ for some $n \in \mathbf{N}\cup\{0\}$, then for any $\mathbf{x}\in[\mathbf{c},\mathbf{d}]\subseteq[\mathbf{a},\mathbf{b}]$ we have

$$\mathcal{O}(\mathbf{x},f) = \sum_{|\boldsymbol{\alpha}|=n+1}\frac{n+1}{\boldsymbol{\alpha}!}\int_{\mathbf{a}}^{\mathbf{b}} K_m(\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}}\cdot\left\{\int_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(1-t)^n dt\right\}d\mathbf{y}. \tag{3.7}$$

Corollary 4. *(i) Let $f:[\mathbf{a},\mathbf{d}]\to\mathbf{R}$ be such that $f\in C^n([\mathbf{a},\mathbf{d}])$ for some $n\in\mathbf{N}$. If $[\mathbf{c},\mathbf{b}]=[\mathbf{a},\mathbf{b}]\cap[\mathbf{c},\mathbf{d}]$, then for every $\mathbf{x}\in[\mathbf{c},\mathbf{b}]$ we have*

$$\begin{aligned}\mathcal{O}(\mathbf{x},f) &= \sum_{|\boldsymbol{\alpha}|=n}\frac{n}{\boldsymbol{\alpha}!}\int_{\mathbf{a}}^{\mathbf{d}} M_m(\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} \\ &\cdot \left\{\int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))-f_{\boldsymbol{\alpha}}(\mathbf{x})](1-t)^{n-1}dt\right\}d\mathbf{y},\end{aligned} \tag{3.8}$$

where

$$M_m = \begin{cases} \frac{1}{\prod_{i=1}^m (b_i-a_i)}, & \text{if } \mathbf{y}\in[\mathbf{a},\mathbf{c}], \\ \frac{1}{\prod_{i=1}^m (b_i-a_i)} - \frac{1}{\prod_{i=1}^m (d_i-c_i)}, & \text{if } \mathbf{y}\in(\mathbf{c},\mathbf{b}], \\ -\frac{1}{\prod_{i=1}^m (d_i-c_i)}, & \text{if } \mathbf{y}\in(\mathbf{b},\mathbf{d}].\end{cases} \tag{3.9}$$

(ii) If $f\in C^{n+1}([\mathbf{a},\mathbf{b}])$ for some $n\in\mathbf{N}\cup\{0\}$, then for any $\mathbf{x}\in[\mathbf{c},\mathbf{b}]=[\mathbf{a},\mathbf{b}]\cap[\mathbf{c},\mathbf{d}]$ we have

$$\mathcal{O}(\mathbf{x},f) = \sum_{|\boldsymbol{\alpha}|=n+1}\frac{n+1}{\boldsymbol{\alpha}!}\int_{\mathbf{a}}^{\mathbf{b}} M_m(\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}}\cdot\left\{\int_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(1-t)^n dt\right\}d\mathbf{y}. \tag{3.10}$$

4. Estimations of the difference of two integral means involving functions of class $C^n([\mathbf{a},\mathbf{b}])$

4.1. **Case $[\mathbf{c},\mathbf{d}]\subseteq[\mathbf{a},\mathbf{b}]$.** We use the integral identity (3.5) to obtain some bounds on the quantity $|\mathcal{O}(\mathbf{x},f)|$.

Theorem 7. *Let* $f : [\mathbf{a}, \mathbf{b}] \to \mathbf{R}$ *be such that for some* $n \in \mathbf{N}$, $f \in C^n([\mathbf{a}, \mathbf{b}])$. *For any* $\mathbf{x} \in [\mathbf{c}, \mathbf{d}] \subseteq [\mathbf{a}, \mathbf{b}]$ *we have*

$$
\begin{aligned}
|\mathcal{O}(\mathbf{x}, f)| &\le \frac{\max_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|=n} \|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty}{n!} \\
&\cdot \left[\prod_{i=1}^m \frac{c_i - a_i}{b_i - a_i} \|\mathbf{x} - \mathbf{a}\|_1^n + \prod_{i=1}^m \frac{b_i - d_i}{b_i - a_i} \|\mathbf{b} - \mathbf{x}\|_1^n \right. \qquad (4.1) \\
&+ \left. \frac{\prod_{i=1}^m (b_i - a_i) - \prod_{i=1}^m (d_i - c_i)}{\prod_{i=1}^m (b_i - a_i)} \left\| \frac{\mathbf{d} - \mathbf{c}}{2} + \left| \mathbf{x} - \frac{\mathbf{c} + \mathbf{d}}{2} \right| \right\|_1^n \right].
\end{aligned}
$$

Proof. For fixed $\mathbf{x} \in [\mathbf{c}, \mathbf{d}]$ and for any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, $\alpha_i \in \mathbf{N} \cup \{0\}$, $i = 1, \ldots, m$, such that $|\boldsymbol{\alpha}| = n$, we have

$$
\begin{aligned}
&\left| \int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})](1 - t)^{n-1} dt \right| \\
\le\ & \|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty \int_0^1 (1 - t)^{n-1} dt = \frac{1}{n} \|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty.
\end{aligned}
$$

So, from (3.5) we get estimation

$$
\begin{aligned}
|\mathcal{O}(\mathbf{x}, f)| &\le \sum_{|\boldsymbol{\alpha}|=n} \frac{\|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty}{\boldsymbol{\alpha}!} \int_{\mathbf{a}}^{\mathbf{b}} |K_m| |\mathbf{y} - \mathbf{x}|^{\boldsymbol{\alpha}} d\mathbf{y} \\
&\le \frac{\max_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|=n} \|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty}{n!} \int_{\mathbf{a}}^{\mathbf{b}} |K_m| \left(\sum_{|\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}} |\mathbf{y} - \mathbf{x}|^{\boldsymbol{\alpha}} \right) d\mathbf{y}, \\
&\le \frac{\max_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|=n} \|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty}{n!} \int_{\mathbf{a}}^{\mathbf{b}} |K_m| \, \|\mathbf{y} - \mathbf{x}\|_1^n \, d\mathbf{y},
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_{\mathbf{a}}^{\mathbf{b}} |K_m| \, \|\mathbf{y} - \mathbf{x}\|_1^n \, d\mathbf{y} = \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{\mathbf{a}}^{\mathbf{c}} \|\mathbf{y} - \mathbf{x}\|_1^n \, d\mathbf{y} \\
+\ & \frac{\prod_{i=1}^m (b_i - a_i) - \prod_{i=1}^m (d_i - c_i)}{\prod_{i=1}^m (b_i - a_i)(d_i - c_i)} \int_{\mathbf{c}}^{\mathbf{d}} \|\mathbf{y} - \mathbf{x}\|_1^n \, d\mathbf{y} + \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{\mathbf{d}}^{\mathbf{b}} \|\mathbf{y} - \mathbf{x}\|_1^n \, d\mathbf{y} \\
\le\ & \max_{\mathbf{y} \in [\mathbf{a}, \mathbf{c}]} \|\mathbf{y} - \mathbf{x}\|_1^n \prod_{i=1}^m \frac{c_i - a_i}{b_i - a_i} + \max_{\mathbf{y} \in [\mathbf{c}, \mathbf{d}]} \|\mathbf{y} - \mathbf{x}\|_1^n \frac{\prod_{i=1}^m (b_i - a_i) - \prod_{i=1}^m (d_i - c_i)}{\prod_{i=1}^m (b_i - a_i)} \\
+\ & \max_{\mathbf{y} \in [\mathbf{d}, \mathbf{b}]} \|\mathbf{y} - \mathbf{x}\|_1^n \prod_{i=1}^m \frac{b_i - d_i}{b_i - a_i}.
\end{aligned}
$$

Further, $\max_{\mathbf{y} \in [\mathbf{a}, \mathbf{c}]} \|\mathbf{y} - \mathbf{x}\|_1^n = \|\mathbf{x} - \mathbf{a}\|_1^n$ and $\max_{\mathbf{y} \in [\mathbf{d}, \mathbf{b}]} \|\mathbf{y} - \mathbf{x}\|_1^n = \|\mathbf{b} - \mathbf{x}\|_1^n$. Also, for $\mathbf{y} \in [\mathbf{c}, \mathbf{d}]$

$$
\begin{aligned}
\|\mathbf{y} - \mathbf{x}\|_1 &= \sum_{i=1}^m |y_i - x_i| \le \sum_{i=1}^m \max\{x_i - c_i, d_i - x_i\} \\
&= \sum_{i=1}^m \left(\frac{d_i - c_i}{2} + \left| x_i - \frac{c_i + d_i}{2} \right| \right) = \left\| \frac{\mathbf{d} - \mathbf{c}}{2} + \left| \mathbf{x} - \frac{\mathbf{c} + \mathbf{d}}{2} \right| \right\|_1.
\end{aligned}
$$

If we define $\mathbf{y_0} = (y_{01}, \ldots, y_{0m}) \in \mathbf{R}^m$ by

$$y_{0i} := \begin{cases} c_i & \text{if} \quad x_i - c_i \geq d_i - x_i, \\ d_i & \text{if} \quad x_i - c_i < d_i - x_i, \end{cases}$$

then $\mathbf{y_0} \in [\mathbf{c}, \mathbf{d}]$ and it is easy to check that $\|\mathbf{y_0} - \mathbf{x}\|_1 = \|((\mathbf{d} - \mathbf{c})/2) + |\mathbf{x} - ((\mathbf{c} + \mathbf{d})/2)\|_1$. We conclude that

$$\max_{\mathbf{y} \in [\mathbf{c}, \mathbf{d}]} \|\mathbf{y} - \mathbf{x}\|_1 = \left\| \frac{\mathbf{d} - \mathbf{c}}{2} + \left| \mathbf{x} - \frac{\mathbf{c} + \mathbf{d}}{2} \right| \right\|_1.$$

So, the proof is completed. □

Corollary 5. *Under the assumptions of Theorem 7 and when* $\mathbf{c} = \mathbf{d} = \mathbf{x}$ *as a limit case, the following inequalities hold*

$$\begin{aligned} & \left| \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{y}) d\mathbf{y} - f(\mathbf{x}) \right. \\ - & \left. \sum_{j=1}^n \sum_{|\boldsymbol{\alpha}|=j} \frac{f_{\boldsymbol{\alpha}}(\mathbf{x})}{\boldsymbol{\alpha}!} \prod_{i=1}^m \frac{(b_i - x_i)^{\alpha_i + 1} + (-1)^{\alpha_i} (x_i - a_i)^{\alpha_i + 1}}{(b_i - a_i)(\alpha_i + 1)} \right| \\ \leq & \frac{\max_{\boldsymbol{\alpha}: |\boldsymbol{\alpha}| = n} \|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty}{n!} \\ \cdot & \left[\prod_{i=1}^m \frac{x_i - a_i}{b_i - a_i} \|\mathbf{x} - \mathbf{a}\|_1^n + \prod_{i=1}^m \frac{b_i - x_i}{b_i - a_i} \|\mathbf{b} - \mathbf{x}\|_1^n \right] \\ \leq & \frac{\max_{\boldsymbol{\alpha}: |\boldsymbol{\alpha}| = n} \|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty}{n!} \left\| \frac{\mathbf{b} - \mathbf{a}}{2} + \left| \mathbf{x} - \frac{\mathbf{a} + \mathbf{b}}{2} \right| \right\|_1^n. \end{aligned} \tag{4.2}$$

Proof. If we put $\mathbf{c} = \mathbf{d} = \mathbf{x}$ in (4.1) we get the first inequality in (4.2). Further,

$$\begin{aligned} \|\mathbf{x} - \mathbf{a}\|_1 &= \sum_{i=1}^m (x_i - a_i) \leq \sum_{i=1}^m \max\{x_i - a_i, b_i - x_i\} \\ &= \sum_{i=1}^m \left(\frac{b_i - a_i}{2} + \left| x_i - \frac{a_i + b_i}{2} \right| \right) = \left\| \frac{\mathbf{b} - \mathbf{a}}{2} + \left| \mathbf{x} - \frac{\mathbf{a} + \mathbf{b}}{2} \right| \right\|_1 \end{aligned}$$

and similar $\|\mathbf{b} - \mathbf{x}\|_1 \leq \left\| \frac{\mathbf{b} - \mathbf{a}}{2} + \left| \mathbf{x} - \frac{\mathbf{a} + \mathbf{b}}{2} \right| \right\|_1$. So, we have

$$\begin{aligned} & \prod_{i=1}^m \frac{x_i - a_i}{b_i - a_i} \|\mathbf{x} - \mathbf{a}\|_1^n + \prod_{i=1}^m \frac{b_i - x_i}{b_i - a_i} \|\mathbf{b} - \mathbf{x}\|_1^n \\ \leq & \frac{\prod_{i=1}^m (x_i - a_i) + \prod_{i=1}^m (b_i - x_i)}{\prod_{i=1}^m (b_i - a_i)} \left\| \frac{\mathbf{b} - \mathbf{a}}{2} + \left| \mathbf{x} - \frac{\mathbf{a} + \mathbf{b}}{2} \right| \right\|_1^n. \end{aligned}$$

Because it is obvious that $\prod_{i=1}^m (x_i - a_i) + \prod_{i=1}^m (b_i - x_i) \leq \prod_{i=1}^m (b_i - a_i)$, we proved the second inequality, too. □

Remark 4. *The second inequality from the above corollary is proved in* [6].

We proceed with some estimations which can be obtained when f satisfies the $\mathbf{r}$-Hölder condition (1.2).

Theorem 8. *Let $f : [\mathbf{a},\mathbf{b}] \to \mathbf{R}$ be such that for some $n \in \mathbf{N}$, $f \in C^n([\mathbf{a},\mathbf{b}])$. Suppose $f_{\boldsymbol{\alpha}}$ satisfies condition (1.2) for all $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m)$, $\alpha_i \in \mathbf{N}\cup\{0\}$, $i = 1,\dots,m$, such that $|\boldsymbol{\alpha}| = n$. Then for any $\mathbf{x} \in [\mathbf{c},\mathbf{d}] \subseteq [\mathbf{a},\mathbf{b}]$ we have*

$$\begin{aligned} |\mathcal{O}(\mathbf{x},f)| &\le \sum_{i=1}^m \frac{L_i\Gamma(r_i+1)}{\Gamma(r_i+1+n)} \left[\prod_{j=1}^m \frac{c_j-a_j}{b_j-a_j} \|\mathbf{x}-\mathbf{a}\|_1^n (x_i-a_i)^{r_i} \right. \\ &+ \frac{\prod_{j=1}^m (b_j-a_j) - \prod_{j=1}^m (d_j-c_j)}{\prod_{j=1}^m (b_j-a_j)} \left\| \frac{\mathbf{d}-\mathbf{c}}{2} + \left|\mathbf{x} - \frac{\mathbf{c}+\mathbf{d}}{2}\right| \right\|_1^n (\max\{x_i-c_i, d_i-x_i\})^{r_i} \\ &+ \left. \prod_{j=1}^m \frac{b_j-d_j}{b_j-a_j} \|\mathbf{b}-\mathbf{x}\|_1^n (b_i-x_i)^{r_i} \right]. \end{aligned} \tag{4.3}$$

Proof. If $f_{\boldsymbol{\alpha}}$ satisfies condition (1.2), then

$$|f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})| \le \sum_{i=1}^m L_i |t(y_i-x_i)|^{r_i} = \sum_{i=1}^m L_i t^{r_i} |y_i-x_i|^{r_i},$$

for any two $\mathbf{x},\mathbf{y} \in [\mathbf{a},\mathbf{b}]$ and for any $t \in [0,1]$. This implies

$$\begin{aligned} &\left| \int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})](1-t)^{n-1} dt \right| \\ \le\ & \sum_{i=1}^m L_i |y_i-x_i|^{r_i} \int_0^1 t^{r_i}(1-t)^{n-1} dt = \sum_{i=1}^m L_i |y_i-x_i|^{r_i} B(r_i+1,n), \end{aligned}$$

where $B(u,v) := \int_0^1 t^{u-1}(1-t)^{v-1}dt$, $u > 0$, $v > 0$ is a beta function. We know that $B(u,v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$, where Γ is a gamma function. Also $\Gamma(n) = (n-1)!$, $n \in \mathbf{N}$ so that

$$\left| \int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})](1-t)^{n-1} dt \right| \le (n-1)! \sum_{i=1}^m \frac{L_i\Gamma(r_i+1)}{\Gamma(r_i+1+n)} |y_i-x_i|^{r_i}.$$

Using this estimation we get from (3.5)

$$\begin{aligned} |\mathcal{O}(\mathbf{x},f)| &\le \sum_{|\boldsymbol{\alpha}|=n} \frac{n}{\boldsymbol{\alpha}!} \int_{\mathbf{a}}^{\mathbf{b}} |K_m| |\mathbf{y}-\mathbf{x}|^{\boldsymbol{\alpha}} \cdot \left((n-1)! \sum_{i=1}^m \frac{L_i\Gamma(r_i+1)}{\Gamma(r_i+1+n)} |y_i-x_i|^{r_i} \right) d\mathbf{y} \\ &= \sum_{i=1}^m \frac{L_i\Gamma(r_i+1)}{\Gamma(r_i+1+n)} \int_{\mathbf{a}}^{\mathbf{b}} \left(\sum_{|\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!} |\mathbf{y}-\mathbf{x}|^{\boldsymbol{\alpha}} \right) |K_m| |y_i-x_i|^{r_i} d\mathbf{y} \\ &= \sum_{i=1}^m \frac{L_i\Gamma(r_i+1)}{\Gamma(r_i+1+n)} \int_{\mathbf{a}}^{\mathbf{b}} |K_m| \|\mathbf{y}-\mathbf{x}\|_1^n |y_i-x_i|^{r_i} d\mathbf{y}, \end{aligned}$$

where as in Theorem 7 we have

$$\begin{aligned}
&\int_{\mathbf{a}}^{\mathbf{b}} |K_m| \|\mathbf{y}-\mathbf{x}\|_1^n |y_i - x_i|^{r_i} d\mathbf{y} \le \max_{\mathbf{y}\in[\mathbf{a},\mathbf{c}]} \|\mathbf{y}-\mathbf{x}\|_1^n |y_i - x_i|^{r_i} \prod_{i=1}^m \frac{c_i - a_i}{b_i - a_i} \\
+ \quad & \max_{\mathbf{y}\in[\mathbf{c},\mathbf{d}]} \|\mathbf{y}-\mathbf{x}\|_1^n |y_i - x_i|^{r_i} \frac{\prod_{i=1}^m (b_i - a_i) - \prod_{i=1}^m (d_i - c_i)}{\prod_{i=1}^m (b_i - a_i)} \\
+ \quad & \max_{\mathbf{y}\in[\mathbf{d},\mathbf{b}]} \|\mathbf{y}-\mathbf{x}\|_1^n |y_i - x_i|^{r_i} \prod_{i=1}^m \frac{b_i - d_i}{b_i - a_i} = \prod_{j=1}^m \frac{c_j - a_j}{b_j - a_j} \|\mathbf{x}-\mathbf{a}\|_1^n (x_i - a_i)^{r_i} \\
+ \quad & \frac{\prod_{j=1}^m (b_j - a_j) - \prod_{j=1}^m (d_j - c_j)}{\prod_{j=1}^m (b_j - a_j)} \left\| \frac{\mathbf{d}-\mathbf{c}}{2} + \left| \mathbf{x} - \frac{\mathbf{c}+\mathbf{d}}{2} \right| \right\|_1^n (\max\{x_i - c_i, d_i - x_i\})^{r_i} \\
+ \quad & \prod_{j=1}^m \frac{b_j - d_j}{b_j - a_j} \|\mathbf{b}-\mathbf{x}\|_1^n (b_i - x_i)^{r_i}.
\end{aligned}$$

So, we proved the theorem. □

Now we will give one special case of this theorem.

Corollary 6. *Let* $f : [\mathbf{a},\mathbf{b}] \to \mathbf{R}$ *be such that for some* $n \in \mathbf{N}$, $f \in C^n([\mathbf{a},\mathbf{b}])$. *Suppose* $f_{\boldsymbol{\alpha}}$ *satisfies condition* $|f_{\boldsymbol{\alpha}}(\mathbf{u}) - f_{\boldsymbol{\alpha}}(\mathbf{v})| \le L\|\mathbf{u}-\mathbf{v}\|_1^r$, *for some* $L > 0$, $r > 0$, *for all* $\mathbf{u},\mathbf{v} \in [\mathbf{a},\mathbf{b}]$ *and for all* $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m)$, $\alpha_i \in \mathbf{N}\cup\{0\}$, $i = 1,\dots,m$, *such that* $|\boldsymbol{\alpha}| = n$. *Then for any* $\mathbf{x} \in [\mathbf{c},\mathbf{d}] \subseteq [\mathbf{a},\mathbf{b}]$ *we have*

$$\begin{aligned}
|\mathcal{O}(\mathbf{x},f)| \quad \le \quad & \frac{L\Gamma(r+1)}{\Gamma(r+1+n)} \left[\prod_{j=1}^m \frac{c_j - a_j}{b_j - a_j} \|\mathbf{x}-\mathbf{a}\|_1^{n+r} + \prod_{j=1}^m \frac{b_j - d_j}{b_j - a_j} \|\mathbf{b}-\mathbf{x}\|_1^{n+r} \right. \\
+ \quad & \left. \frac{\prod_{j=1}^m (b_j - a_j) - \prod_{j=1}^m (d_j - c_j)}{\prod_{j=1}^m (b_j - a_j)} \left\| \frac{\mathbf{d}-\mathbf{c}}{2} + \left| \mathbf{x} - \frac{\mathbf{c}+\mathbf{d}}{2} \right| \right\|_1^{n+r} \right]
\end{aligned} \tag{4.4}$$

Proof. For any two $\mathbf{x},\mathbf{y} \in [\mathbf{a},\mathbf{b}]$ and for any $t \in [0,1]$, $f_{\boldsymbol{\alpha}}$ satisfies the condition

$$|f_{\boldsymbol{\alpha}}(\mathbf{x} + t(\mathbf{y}-\mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})| \le L\|t(\mathbf{y}-\mathbf{x})\|_1^r = Lt^r \|\mathbf{y}-\mathbf{x}\|_1^r.$$

This implies

$$\begin{aligned}
& \left| \int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x} + t(\mathbf{y}-\mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})](1-t)^{n-1} dt \right| \le L\|\mathbf{y}-\mathbf{x}\|_1^r \int_0^1 t^r (1-t)^{n-1} dt \\
= \quad & L\|\mathbf{y}-\mathbf{x}\|_1^r B(r+1,n) = \frac{L\Gamma(r+1)(n-1)!}{\Gamma(r+1+n)} \|\mathbf{y}-\mathbf{x}\|_1^r.
\end{aligned}$$

The rest of the proof is similar as the proof of the Theorem 8. □

Remark 5. *It should be noted that inequalities (4.3) and (4.4) are valid for $n = 0$ too. When $n = 0$, (4.3) becomes*

$$\begin{aligned} & \left| \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{\mathbf{a}}^{\mathbf{b}} f(y)dy - \frac{1}{\prod_{i=1}^m (d_i - c_i)} \int_{\mathbf{c}}^{\mathbf{d}} f(y)dy \right| \\ \leq \; & \sum_{i=1}^m L_i \left[\prod_{j=1}^m \frac{c_j - a_j}{b_j - a_j} (x_i - a_i)^{r_i} + \prod_{j=1}^m \frac{b_j - d_j}{b_j - a_j} (b_i - x_i)^{r_i} \right. \\ + \; & \left. \frac{\prod_{j=1}^m (b_j - a_j) - \prod_{j=1}^m (d_j - c_j)}{\prod_{j=1}^m (b_j - a_j)} (\max\{x_i - c_i, d_i - x_i\})^{r_i} \right] \end{aligned} \tag{4.5}$$

and (4.4) becomes

$$\begin{aligned} & \left| \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{\mathbf{a}}^{\mathbf{b}} f(y)dy - \frac{1}{\prod_{i=1}^m (d_i - c_i)} \int_{\mathbf{c}}^{\mathbf{d}} f(y)dy \right| \\ \leq \; & L \left[\prod_{j=1}^m \frac{c_j - a_j}{b_j - a_j} \|\mathbf{x} - \mathbf{a}\|_1^r + \prod_{j=1}^m \frac{b_j - d_j}{b_j - a_j} \|\mathbf{b} - \mathbf{x}\|_1^r \right. \\ + \; & \left. \frac{\prod_{j=1}^m (b_j - a_j) - \prod_{j=1}^m (d_j - c_j)}{\prod_{j=1}^m (b_j - a_j)} \left\| \frac{\mathbf{d} - \mathbf{c}}{2} + \left| \mathbf{x} - \frac{\mathbf{c} + \mathbf{d}}{2} \right| \right\|_1^r \right]. \end{aligned} \tag{4.6}$$

4.2. **Case $[\mathbf{c}, \mathbf{b}] = [\mathbf{a}, \mathbf{b}] \cap [\mathbf{c}, \mathbf{d}]$.** We use the integral identity (3.8) to obtain some bounds on the quantity $|\mathcal{O}(\mathbf{x}, f)|$.

Theorem 9. *Let $f : [\mathbf{a}, \mathbf{d}] \to \mathbf{R}$ be such that for some $n \in \mathbf{N}$, $f \in C^n([\mathbf{a}, \mathbf{d}])$. For any $\mathbf{x} \in [\mathbf{c}, \mathbf{b}] = [\mathbf{a}, \mathbf{b}] \cap [\mathbf{c}, \mathbf{d}]$ we have*

$$\begin{aligned} & |\mathcal{O}(\mathbf{x}, f)| \leq \frac{\max_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|=n} \|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty}{n!} \\ \cdot \; & \left[\prod_{i=1}^m \frac{c_i - a_i}{b_i - a_i} \|\mathbf{x} - \mathbf{a}\|_1^n + \prod_{i=1}^m \frac{d_i - b_i}{d_i - c_i} \|\mathbf{d} - \mathbf{x}\|_1^n \right. \\ + \; & \left. \frac{\prod_{i=1}^m (b_i - c_i) \left(\prod_{i=1}^m (b_i - a_i) - \prod_{i=1}^m (d_i - c_i) \right)}{\prod_{i=1}^m (b_i - a_i)(d_i - c_i)} \left\| \frac{\mathbf{b} - \mathbf{c}}{2} + \left| \mathbf{x} - \frac{\mathbf{c} + \mathbf{b}}{2} \right| \right\|_1^n \right]. \end{aligned} \tag{4.7}$$

Proof. Similar as proof of the Theorem 7. □

We proceed with some estimations which can be obtained when f satisfies the $\mathbf{r}$-Hölder condition (1.2).

Theorem 10. *Let $f : [\mathbf{a}, \mathbf{d}] \to \mathbf{R}$ be such that for some $n \in \mathbf{N}$, $f \in C^n([\mathbf{a}, \mathbf{d}])$. Suppose $f_{\boldsymbol{\alpha}}$ satisfies condition (1.2) for all $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, $\alpha_i \in \mathbf{N} \cup \{0\}$, $i =$*

$1,\dots,m$, *such that* $|\boldsymbol{\alpha}|=n$. *Then for any* $\mathbf{x}\in[\mathbf{c},\mathbf{b}]=[\mathbf{a},\mathbf{b}]\cap[\mathbf{c},\mathbf{d}]$ *we have*

$$
\begin{aligned}
|\mathcal{O}(\mathbf{x},f)| &\le \sum_{i=1}^{m}\frac{L_i\Gamma(r_i+1)}{\Gamma(r_i+1+n)}\left[\prod_{j=1}^{m}\frac{c_j-a_j}{b_j-a_j}\|\mathbf{x}-\mathbf{a}\|_1^n(x_i-a_i)^{r_i}\right.\\
&+ \frac{\prod_{j=1}^{m}(b_j-c_j)\left(\prod_{j=1}^{m}(b_j-a_j)-\prod_{j=1}^{m}(d_j-c_j)\right)}{\prod_{j=1}^{m}(b_j-a_j)(d_j-c_j)}\\
&\cdot \left\|\frac{\mathbf{b}-\mathbf{c}}{2}+\left|\mathbf{x}-\frac{\mathbf{c}+\mathbf{b}}{2}\right|\right\|_1^n(\max\{x_i-c_i,b_i-x_i\})^{r_i}\\
&+ \left.\prod_{j=1}^{m}\frac{d_j-b_j}{d_j-c_j}\|\mathbf{d}-\mathbf{x}\|_1^n(d_i-x_i)^{r_i}\right]. \qquad (4.8)
\end{aligned}
$$

Proof. Similar as proof of the Theorem 8. □

Now we will give one special case of this theorem.

Corollary 7. *Let* $f:[\mathbf{a},\mathbf{d}]\to\mathbf{R}$ *be such that for some* $n\in\mathbf{N}$, $f\in C^n([\mathbf{a},\mathbf{d}])$. *Suppose* $f_{\boldsymbol{\alpha}}$ *satisfies condition* $|f_{\boldsymbol{\alpha}}(\mathbf{u})-f_{\boldsymbol{\alpha}}(\mathbf{v})|\le L\|\mathbf{u}-\mathbf{v}\|_1^r$, *for some* $L>0$, $r>0$, *for all* $\mathbf{u},\mathbf{v}\in[\mathbf{a},\mathbf{b}]$ *and for all* $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_m)$, $\alpha_i\in\mathbf{N}\cup\{0\}$, $i=1,\dots,m$, *such that* $|\boldsymbol{\alpha}|=n$. *Then for any* $\mathbf{x}\in[\mathbf{c},\mathbf{b}]=[\mathbf{a},\mathbf{b}]\cap[\mathbf{c},\mathbf{d}]$ *we have*

$$
\begin{aligned}
|\mathcal{O}(\mathbf{x},f)| &\le \frac{L\Gamma(r+1)}{\Gamma(r+1+n)}\left[\prod_{j=1}^{m}\frac{c_j-a_j}{b_j-a_j}\|\mathbf{x}-\mathbf{a}\|_1^{n+r}+\prod_{j=1}^{m}\frac{d_j-b_j}{d_j-c_j}\|\mathbf{d}-\mathbf{x}\|_1^{n+r}\right.\\
&+ \frac{\prod_{j=1}^{m}(b_j-c_j)\left(\prod_{j=1}^{m}(b_j-a_j)-\prod_{j=1}^{m}(d_j-c_j)\right)}{\prod_{j=1}^{m}(b_j-a_j)(d_j-c_j)}\\
&\cdot \left.\left\|\frac{\mathbf{b}-\mathbf{c}}{2}+\left|\mathbf{x}-\frac{\mathbf{c}+\mathbf{b}}{2}\right|\right\|_1^{n+r}\right]. \qquad (4.9)
\end{aligned}
$$

Proof. Similar as proof of the Corollary 6. □

Remark 6. *It should be noted that inequalities (4.8) and (4.9) are valid for* $n=0$ *too. When* $n=0$, *(4.8) becomes*

$$
\begin{aligned}
&\left|\frac{1}{\prod_{i=1}^{m}(b_i-a_i)}\int_{\mathbf{a}}^{\mathbf{b}}f(y)dy-\frac{1}{\prod_{i=1}^{m}(d_i-c_i)}\int_{\mathbf{c}}^{\mathbf{d}}f(y)dy\right|\\
\le\ & \sum_{i=1}^{m}L_i\left[\prod_{j=1}^{m}\frac{c_j-a_j}{b_j-a_j}(x_i-a_i)^{r_i}+\prod_{j=1}^{m}\frac{d_j-b_j}{d_j-c_j}(d_i-x_i)^{r_i}\right. \qquad (4.10)\\
+\ & \left.\frac{\prod_{j=1}^{m}(b_j-c_j)\left(\prod_{j=1}^{m}(b_j-a_j)-\prod_{j=1}^{m}(d_j-c_j)\right)}{\prod_{j=1}^{m}(b_j-a_j)(d_j-c_j)}(\max\{x_i-c_i,b_i-x_i\})^{r_i}\right]
\end{aligned}
$$

and (4.9) becomes

$$
\begin{aligned}
&\left|\frac{1}{\prod_{i=1}^m (b_i-a_i)}\int_{\mathbf{a}}^{\mathbf{b}} f(y)dy - \frac{1}{\prod_{i=1}^m (d_i-c_i)}\int_{\mathbf{c}}^{\mathbf{d}} f(y)dy\right| \\
\leq\ & L\left[\prod_{j=1}^m \frac{c_j-a_j}{b_j-a_j}\|\mathbf{x}-\mathbf{a}\|_1^r + \prod_{j=1}^m \frac{d_j-b_j}{d_j-c_j}\|\mathbf{d}-\mathbf{x}\|_1^r\right. \qquad (4.11)\\
+\ & \left.\frac{\prod_{j=1}^m (b_j-c_j)\left(\prod_{j=1}^m (b_j-a_j)-\prod_{j=1}^m (d_j-c_j)\right)}{\prod_{j=1}^m (b_j-a_j)(d_j-c_j)}\left\|\frac{\mathbf{b}-\mathbf{c}}{2}+\left|\mathbf{x}-\frac{\mathbf{c}+\mathbf{b}}{2}\right|\right\|_1^r\right].
\end{aligned}
$$

5. Estimations of the difference of two integral means involving functions of class $C^{n+1}([\mathbf{a},\mathbf{b}])$

5.1. **Case $[\mathbf{c},\mathbf{d}]\subseteq[\mathbf{a},\mathbf{b}]$.** We use the integral identity (3.7) to obtain some bounds on the quantity $|\mathcal{O}(\mathbf{x},f)|$.

Theorem 11. *Let $f:[\mathbf{a},\mathbf{b}]\to\mathbf{R}$ be such that for some $n\in\mathbf{N}$, $f\in C^{n+1}([\mathbf{a},\mathbf{b}])$. For any $\mathbf{x}\in[\mathbf{c},\mathbf{d}]\subseteq[\mathbf{a},\mathbf{b}]$ we have*

$$
\begin{aligned}
|\mathcal{O}(\mathbf{x},f)| \leq\ & \frac{\max_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|=n+1}\|f_{\boldsymbol{\alpha}}\|_\infty}{(n+1)!} \\
\cdot\ & \left[\prod_{i=1}^m \frac{c_i-a_i}{b_i-a_i}\|\mathbf{x}-\mathbf{a}\|_1^{n+1} + \prod_{i=1}^m \frac{b_i-d_i}{b_i-a_i}\|\mathbf{b}-\mathbf{x}\|_1^{n+1}\right. \qquad (5.1)\\
+\ & \left.\frac{\prod_{i=1}^m (b_i-a_i)-\prod_{i=1}^m (d_i-c_i)}{\prod_{i=1}^m (b_i-a_i)}\left\|\frac{\mathbf{d}-\mathbf{c}}{2}+\left|\mathbf{x}-\frac{\mathbf{c}+\mathbf{d}}{2}\right|\right\|_1^{n+1}\right].
\end{aligned}
$$

Proof. If $f\in C^{n+1}([\mathbf{a},\mathbf{b}])$ for some $n\in\mathbf{N}\cup\{0\}$, then for any partial derivative $f_{\boldsymbol{\alpha}}$ with $|\boldsymbol{\alpha}|=n+1$ and for any $\mathbf{x}\in[\mathbf{c},\mathbf{d}]$

$$
\left|\int_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(1-t)^n dt\right| \leq \|f_{\boldsymbol{\alpha}}\|_\infty \int_0^1 (1-t)^n dt = \frac{\|f_{\boldsymbol{\alpha}}\|_\infty}{n+1}.
$$

Using this estimation, from (3.7), similar as in Theorem 7 we get inequality (5.1). □

Corollary 8. *Let the assumptions of Theorem 11 be satisfied. Additionally, suppose that, for some $\mathbf{x}\in[\mathbf{c},\mathbf{d}]$ all partial derivatives $f_{\boldsymbol{\alpha}}$, $1\leq|\boldsymbol{\alpha}|\leq n$ fulfill $f_{\boldsymbol{\alpha}}(\mathbf{x})=0$. Then we have*

$$
\begin{aligned}
&\left|\frac{1}{\prod_{i=1}^m (b_i-a_i)}\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{y})d\mathbf{y} - \frac{1}{\prod_{i=1}^m (d_i-c_i)}\int_{\mathbf{c}}^{\mathbf{d}} f(\mathbf{y})d\mathbf{y}\right| \\
\leq\ & \frac{\max_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|=n+1}\|f_{\boldsymbol{\alpha}}\|_\infty}{(n+1)!}\left[\prod_{i=1}^m \frac{c_i-a_i}{b_i-a_i}\|\mathbf{x}-\mathbf{a}\|_1^{n+1} + \prod_{i=1}^m \frac{b_i-d_i}{b_i-a_i}\|\mathbf{b}-\mathbf{x}\|_1^{n+1}\right.\\
+\ & \left.\frac{\prod_{i=1}^m (b_i-a_i)-\prod_{i=1}^m (d_i-c_i)}{\prod_{i=1}^m (b_i-a_i)}\left\|\frac{\mathbf{d}-\mathbf{c}}{2}+\left|\mathbf{x}-\frac{\mathbf{c}+\mathbf{d}}{2}\right|\right\|_1^{n+1}\right]. \qquad (5.2)
\end{aligned}
$$

Proof. Since $f_{\boldsymbol{\alpha}}(\mathbf{x}) = 0$ for $1 \le |\boldsymbol{\alpha}| \le n$, we have

$$\mathcal{O}(\mathbf{x}, f) = \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{y}) d\mathbf{y} - \frac{1}{\prod_{i=1}^m (d_i - c_i)} \int_{\mathbf{c}}^{\mathbf{d}} f(\mathbf{y}) d\mathbf{y}$$

and the desired result follows by Theorem 11. □

5.2. **Case $[\mathbf{c}, \mathbf{b}] = [\mathbf{a}, \mathbf{b}] \cap [\mathbf{c}, \mathbf{d}]$.** We use the integral identity (3.10) to obtain some bounds on the quantity $|\mathcal{O}(\mathbf{x}, f)|$.

Theorem 12. *Let $f : [\mathbf{a}, \mathbf{d}] \to \mathbf{R}$ be such that for some $n \in \mathbf{N}$, $f \in C^{n+1}([\mathbf{a}, \mathbf{d}])$. For any $\mathbf{x} \in [\mathbf{c}, \mathbf{b}] = [\mathbf{a}, \mathbf{b}] \cap [\mathbf{c}, \mathbf{d}]$ we have*

$$\begin{aligned} |\mathcal{O}(\mathbf{x}, f)| &\le \frac{\max_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|=n+1} \|f_{\boldsymbol{\alpha}}\|_\infty}{(n+1)!} \\ &\cdot \left[\prod_{i=1}^m \frac{c_i - a_i}{b_i - a_i} \|\mathbf{x} - \mathbf{a}\|_1^{n+1} + \prod_{i=1}^m \frac{d_i - b_i}{d_i - c_i} \|\mathbf{d} - \mathbf{x}\|_1^{n+1} \right. \\ &+ \left. \frac{\prod_{i=1}^m (b_i - c_i) \left(\prod_{i=1}^m (b_i - a_i) - \prod_{i=1}^m (d_i - c_i) \right)}{\prod_{i=1}^m (b_i - a_i)(d_i - c_i)} \left\| \frac{\mathbf{b} - \mathbf{c}}{2} + \left| \mathbf{x} - \frac{\mathbf{c} + \mathbf{b}}{2} \right| \right\|_1^{n+1} \right]. \end{aligned} \tag{5.3}$$

Proof. Similar as proof of the Theorem 11. □

Corollary 9. *Let the assumptions of Theorem 12 be satisfied. Additionally, suppose that, for some $\mathbf{x} \in [\mathbf{c}, \mathbf{b}]$ all partial derivatives $f_{\boldsymbol{\alpha}}$, $1 \le |\boldsymbol{\alpha}| \le n$ fulfill $f_{\boldsymbol{\alpha}}(\mathbf{x}) = 0$. Then we have*

$$\begin{aligned} &\left| \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{y}) d\mathbf{y} - \frac{1}{\prod_{i=1}^m (d_i - c_i)} \int_{\mathbf{c}}^{\mathbf{d}} f(\mathbf{y}) d\mathbf{y} \right| \qquad (5.4) \\ \le\; & \frac{\max_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|=n+1} \|f_{\boldsymbol{\alpha}}\|_\infty}{(n+1)!} \left[\prod_{i=1}^m \frac{c_i - a_i}{b_i - a_i} \|\mathbf{x} - \mathbf{a}\|_1^{n+1} + \prod_{i=1}^m \frac{d_i - b_i}{d_i - c_i} \|\mathbf{d} - \mathbf{x}\|_1^{n+1} \right. \\ +\; & \left. \frac{\prod_{i=1}^m (b_i - c_i) \left(\prod_{i=1}^m (b_i - a_i) - \prod_{i=1}^m (d_i - c_i) \right)}{\prod_{i=1}^m (b_i - a_i)(d_i - c_i)} \left\| \frac{\mathbf{b} - \mathbf{c}}{2} + \left| \mathbf{x} - \frac{\mathbf{c} + \mathbf{b}}{2} \right| \right\|_1^{n+1} \right]. \end{aligned}$$

Proof. Similar as proof of the Corollary 8. □

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Modified Rayleigh Conjecture Method with optimally placed sources

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Abstract.
The Rayleigh conjecture on the representation of the scattered field in the exterior of an obstacle D is widely used in applications. However this conjecture is false for some obstacles. AGR introduced the Modified Rayleigh Conjecture (MRC). In this paper we present a version of the MRC based on an optimal choice of sources. The method is implemented and tested for various 2D and 3D obstacles including a triangle, a cube, and ellipsoids. The MRC method is easy to implement for both simple and complex geometries. It is shown to be a viable alternative to other obstacle scattering methods.

Key words: obstacle scattering, modified Rayleigh conjecture.

Math. Subj. classification: 35J05, 65M99, 78A40

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1 Introduction.

In this paper we present a novel numerical method for Direct Obstacle Scattering Problems based on the Modified Rayleigh Conjecture (MRC). The basic theoretical foundation of the method was developed in [9]. The MRC has the appeal of an easy implementation for obstacles of complicated geometry, e.g. having edges and corners. In our numerical experiments the method has shown itself to be a competitive alternative to the BIEM (boundary integral equations method), see [4]. Also, unlike the BIEM, one can apply the algorithm to different obstacles with very little additional effort.

However, a numerical implementation of the MRC, may be inefficient or fail, so some effort is needed to make it efficient. Our aim is to describe such an implementation, and to show the advantages of a properly implemented MRC method over BIEM, for example. In this paper we describe an MRC method with an optimal choice of sources, and apply it to various 2D and 3D obstacles including ellipsoids and a cube.

In our previous paper [4] we described another implementation of the MRC. That method (Multi-point MRC) could be used for 2D obstacles of a relatively simple geometry, but it failed for some 2D obstacles, and it was not successful for 3D problems. The difficulty was in excessive demands on computing resources. The new version presented here is an iterative implementation of the MRC method. It allows a significant improvement over previously discussed results, and provides an efficient and economical numerical solution for 3D obstacle scattering problems.

We formulate the obstacle scattering problem in a 3D setting with the Dirichlet boundary condition, but the method can also be used for the Neumann and Robin boundary conditions.

Consider a bounded domain $D \subset \mathbb{R}^3$, with a Lipschitz boundary S. Denote the exterior domain by $D' = \mathbb{R}^3 \backslash D$. Let $\alpha, \alpha' \in S^2$ be unit vectors, and S^2 be the unit sphere in $\mathbb{R}^3$.

The acoustic wave scattering problem by a soft obstacle D consists in finding the (unique) solution to the problem (1)-(2):

$$\left(\nabla^2 + k^2\right) u = 0 \text{ in } D', \quad u = 0 \text{ on } S, \tag{1}$$

$$u = u_0 + A(\alpha', \alpha)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \alpha' := \frac{x}{r}. \tag{2}$$

Here $u_0 := e^{ik\alpha\cdot x}$ is the incident field, $v := u - u_0$ is the scattered field, $A(\alpha', \alpha)$ is the scattering amplitude, its k-dependence is not shown, $k > 0$ is the wavenumber. Denote

$$A_\ell(\alpha) := \int_{S^2} A(\alpha', \alpha)\overline{Y_\ell(\alpha')}d\alpha', \tag{3}$$

where $Y_\ell(\alpha)$ are the orthonormal spherical harmonics, $Y_\ell = Y_{\ell m}, -\ell \le m \le \ell$. Let $h_\ell(r)$ be the spherical Hankel functions, normalized so that $h_\ell(r) \sim \frac{e^{ikr}}{r}$ as $r \to +\infty$, see [8].

Our algorithm for the MRC method for 3D obstacles can be described as follows.

Let z be a point (source) inside the obstacle D. For $x \in D'$, let

$$\alpha' = \frac{x-z}{|x-z|}, \quad \psi_\ell(x,z) = Y_\ell(\alpha')h_\ell(k|x-z|). \tag{4}$$

Let $g_1(s) = u_0(s) = u_0(s,\alpha),\ s \in S$.

Minimize

$$\Phi(z_1, \mathbf{c}(z_1)) := \min_{z\in D} \min_{\mathbf{c}\in\mathbb{C}^N} \|g_1(s) + \sum_{\ell=0}^{L} c_\ell \psi_\ell(s,z)\|_{L^2(S)}, \tag{5}$$

where $\mathbf{c} = \{c_\ell\} = \{c_{\ell m}\}_{0\le\ell\le L, -\ell\le m\le\ell}$, $L \ge 0$ is a fixed integer and $\sum_{\ell=0}^{L} := \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell}$. Let

$$v_1(x) = \sum_{\ell=0}^{L} c_\ell(z_1)\psi_\ell(x,z_1), \quad c_\ell(z_1) = c_\ell(z_1,\alpha). \tag{6}$$

The requirement (5) means that the total field $u(s) = g_1(s) + v_1(s)$ has to be as close to zero as possible on the boundary S, so that it approximates best the Dirichlet boundary condition in (1). This is achieved by varying the interior point $z \in D$ and choosing the coefficients $\mathbf{c}(z) \in \mathbb{C}^N$ giving g_1+v_1 the best fit to zero on the boundary S. Let the minimum in (5) be attained at $z_1 \in D$. If the resulting value of the residual $r^{min} = \Phi(z_1, \mathbf{c}(z_1))$ is smaller than the prescribed tolerance ϵ, than the procedure is finished. The sought approximate scattered field is $v_1(x),\ x \in D'$ (see Theorem 2 below), and the approximate scattering amplitude is

$$A_1(\alpha',\alpha) = e^{-ik\alpha'\cdot z_1} \sum_{\ell=0}^{L} c_\ell(z_1) Y_\ell(\alpha')\,. \tag{7}$$

Note that $c_\ell(z_1) = c_\ell(z_1,\alpha)$.

The expression for $A_1(\alpha',\alpha)$ in (7) is obtained from (6) by letting $|x| \to \infty$ in $x = \alpha'|x|$, because of our normalization

$$h_\ell(k|x|) = \frac{e^{ik|x|}}{|x|}\left\{1 + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \tag{8}$$

and $|x-z| = |x| - \alpha'\cdot z + O(1/|x|)$ as $|x| \to \infty$.

If, on the other hand, the residual $r^{min} > \epsilon$, then we continue by trying to improve on the already obtained fit in (5) as follows. Adjust the field on the boundary by letting $g_2(s) = g_1(s) + v_1(s),\ s \in S$, and do the minimization (5) with $g_2(s)$ instead of $g_1(s)$, etc. Continue with the iterations until the required tolerance ϵ on the boundary S is attained. At the same time keep track of the changing approximate scattered field $v_n(x)$, and the scattering amplitude $A_n(\alpha',\alpha)$. In this construction $g_{n+1} = u_0 + v_n$ on S. The goal of (5) is to

obtain $g_n \to 0$ in $L^2(S)$ as $n \to \infty$, yielding $u_0 + v_n \to 0$ in $L^2(S)$. According to Theorem 2, this gives an approximate scattered solution v_n on D' to (1)-(2).

This iterative feature allows the algorithm to solve scattering problems intractable by previously developed MRC methods (see [4]).

Here is a precise description of the algorithm.

MRC method with optimal choice of sources.

For $z \in D$ and $\ell \geq 0$ functions $\psi_\ell(x, z)$ are defined in (4).

1. **Initialization.** Fix $\epsilon > 0,\ L \geq 0,\ N_{max} > 0$. Let $n = 0,\ v_0(x) = 0,\ A_0(\alpha', \alpha) = 0$, and $g_1(s) = u_0(s),\ s \in S$.

2. **Iteration.**

   (a) Increase the value of n by 1.

   (b) Minimize
   $$\Phi(z_n, \mathbf{c}(z_n)) := \min_{z \in D} \min_{\mathbf{c} \in \mathbb{C}^N} \|g_n(s) + \sum_{\ell=0}^{L} c_\ell \psi_\ell(s, z)\|_{L^2(S)},$$
   with the minimal value attained at $z_n \in D,\ \mathbf{c}(z_n) \in \mathbb{C}^N$.

   (c) Let
   $$v_n(x) = v_{n-1}(x) + \sum_{\ell=0}^{L} c_\ell(z_n)\psi_\ell(x, z_n), \quad x \in D',$$
   $$A_n(\alpha', \alpha) = A_{n-1}(\alpha', \alpha) + e^{-ik\alpha' \cdot z_n} \sum_{\ell=0}^{L} c_\ell(z_n) Y_\ell(\alpha'),$$
   and
   $$g_{n+1}(s) = g_n(s) + \sum_{\ell=0}^{L} c_\ell(z_n)\psi_\ell(s, z_n), \quad s \in S,$$
   that is $g_{n+1}(s) = u_0(s) + v_n(s),\ s \in S$.

   (d) Let
   $$r^{min} := \Phi(z_n, \mathbf{c}(z_n)).$$

3. **Stopping criterion.**

   (a) If $r^{min} \leq \epsilon$, then stop; $v_n(x)$ is the approximate scattered field, and $A_n(\alpha', \alpha)$ is the approximate scattering amplitude.

   (b) If $r^{min} > \epsilon$, and $n < N_{max}$, then repeat the iterative step (2).

   (c) If $r^{min} > \epsilon$, and $n = N_{max}$, then the procedure failed.

2 Direct scattering problems and the Rayleigh conjecture.

Let a ball $B_R := \{x : |x| \leq R\}$ contain the obstacle D. In the region $|x| > R$ the solution to (1)-(2) is:

$$u(x,\alpha) = e^{ik\alpha\cdot x} + \sum_{\ell=0}^{\infty} A_\ell(\alpha)\Psi_\ell(x), \quad \Psi_\ell := Y_\ell(\alpha')h_\ell(k|x|), \quad \alpha' = \frac{x}{|x|}, \tag{9}$$

where the sum includes the summation with respect to m, $-\ell \leq m \leq \ell$, and $A_\ell(\alpha)$ are defined in (3). Note that $\Psi_\ell(x) = \psi_\ell(x, 0)$.

The Rayleigh conjecture (RC) is: the series (9) converges up to the boundary S (originally RC dealt with periodic structures, gratings). This conjecture is false for many obstacles, but is true for some ([1, 2, 6, 10]). For example, if $n = 2$ and D is an ellipse, then the series analogous to (9) converges in the region $|x| > a$, where $2a$ is the distance between the foci of the ellipse [1]. In the engineering literature there are numerical algorithms, based on the Rayleigh conjecture. These algorithms use projection methods and are reported to be unstable. Moreover, no error estimate has been obtained for such algorithms [2, 6]. These algorithms cannot converge for arbitrary obstacles, because the Rayleigh conjecture is false for some obstacles. In contrast, the MRC-based algorithm, proposed here, converges and an error estimate for the approximate solution it yields is obtained. This error estimate is sharp in the order ϵ. We discuss the Dirichlet condition but a similar argument is applicable to the Neumann boundary condition, corresponding to acoustically hard obstacles.

What we call The Modified Rayleigh Conjecture (MRC) is actually the following theorem (see [9]):

Theorem 1. *Let v be the scattered solution to (1)-(2). Then there exists a positive integer $L = L(\epsilon)$ and the coefficients $c_\ell = c_\ell(\epsilon)$, $0 \leq \ell \leq L(\epsilon)$ such that*

(i).

$$||u_0 + v_\epsilon||_{L^2(S)} \leq \epsilon, \tag{10}$$

where

$$v_\epsilon(x) = \sum_{\ell=0}^{L(\epsilon)} c_\ell(\epsilon)\Psi_\ell(x). \tag{11}$$

(ii).

$$\|v_\epsilon - v\|_{L^2(S)} \leq \epsilon \tag{12}$$

and

$$|||v_\epsilon - v||| = O(\epsilon)\,,\ \epsilon \to 0\,, \tag{13}$$

where

$$|||\cdot||| = \|\cdot\|_{H^m_{loc}(D')} + \|\cdot\|_{L^2(D';(1+|x|)^{-\gamma})}\,,$$

$\gamma > 1\,,\ m > 0$ *is an arbitrary integer,* H^m *is the Sobolev space.*

(iii).

$$c_\ell(\epsilon) \to A_\ell\,,\ \textit{as}\ \epsilon \to 0\,,\ \forall \ell\,,$$

where $A_\ell := A_\ell(\alpha)$ *is defined in* (3).

The proof of Theorem 1 is not given here since we need a different version of the MRC method justified in Theorem 2, and the proof of this theorem is given below.

The difference between RC and MRC is: (10) does not hold if one replaces v_ϵ by $\sum_{\ell=0}^{L} A_\ell(\alpha)\Psi_\ell$, and lets $L \to \infty$ (instead of letting $\epsilon \to 0$). Indeed, the series $\sum_{\ell=0}^{\infty} A_\ell(\alpha)\Psi_\ell$ diverges at some points of the boundary for many obstacles. Note also that the coefficients in (11) depend on ϵ, so (11) is *not* a partial sum of a series.

For the Neumann boundary condition one minimizes

$$\left\|\frac{\partial[u_0 + \sum_{\ell=0}^{L} c_\ell \psi_\ell]}{\partial N}\right\|_{L^2(S)}$$

with respect to c_ℓ, and obtains essentially the same results.

See [11] for an extension of these results to scattering by periodic structures.

The construction of the approximate scattered field by the MRC method described in Theorem 1 is not satisfactory from the numerical perspective, since it imposes excessive demands on computational resources. See section 4 for an additional discussion. The MRC method with optimal choice of sources was designed to overcome these difficulties. It is based on the following Theorem:

Theorem 2. *Let v be the scattered solution to (1)-(2). Let $\epsilon > 0$, and L be a nonnegative integer. Suppose U is an open subset of D.*

Then there exist a finite subset $\{z_1, z_2, ..., z_J\} \subset U$ and the coefficients $c_\ell(\epsilon, z_j),\ 0 \le \ell \le L,\ 1 \le j \le J$ such that

(i).

$$||u_0 + v_\epsilon||_{L^2(S)} \le \epsilon\,, \tag{14}$$

where

$$v_\epsilon(x) := \sum_{j=1}^{J}\sum_{\ell=0}^{L} c_\ell(\epsilon, z_j)\psi_\ell(x, z_j) \tag{15}$$

and the functions $\psi_\ell(x, z)$ are defined as in (4).

(ii).

$$\|v_\epsilon - v\|_{L^2(S)} \le \epsilon \tag{16}$$

and

$$|||v_\epsilon - v||| = O(\epsilon)\,, \quad \epsilon \to 0\,, \tag{17}$$

where

$$|||\cdot||| = \|\cdot\|_{H^m_{loc}(D')} + \|\cdot\|_{L^2(D';(1+|x|)^{-\gamma})}\,,$$

$\gamma > 1\,,\ m > 0$ *is an arbitrary integer,* H^m *is the Sobolev space.*

Proof. (i) Let $\{z_j\}_{j=1}^{\infty}$ be a countable dense subset of U. To establish (14) it is sufficient to show that

$$H := \overline{span}\{\psi_\ell(s,z_j) \ : \ 0 \le \ell \le L, \quad j=1,2,...\} = L^2(S)\,. \tag{18}$$

Suppose that there exists $p \in L^2(S),\ p \neq 0$ such that $p \perp H$ in $L^2(S)$. Define the single-layer potential by

$$W(y) = \int_S \frac{e^{ik|x-y|}}{|x-y|} p(x)\ ds(x)\,, \quad y \in \mathbb{R}^3\,. \tag{19}$$

Then

$$W(z_j) = \int_S \psi_0(x,z_j)p(x)\ ds(x) = 0 \tag{20}$$

for $j = 1,2,....$

The continuity of the single-layer potential in $\mathbb{R}^3$ implies that $W(y) = 0$ for all $y \in U$. By the unique continuation principle $W \equiv 0$ in D. In particular $W = 0$ on the boundary S. Since W is an outgoing solution of $(\nabla^2 + k^2)W = 0$, in D' with $W = 0$ on S, one concludes from the uniqueness of solutions to the Dirichlet problem in D' that $W \equiv 0$ in $\mathbb{R}^3$. Finally, the jump properties of the normal derivative of the single-layer potential imply that $p = 0$ in $L^2(S)$.

(ii) Inequality (16) is the same as (14), since $v = -u_0$ on S. Estimate (17) follows from (16) and Lemma 1.

□

Lemma 1. *Given $g \in L_2(S)$, let w be the outgoing solution of the Exterior Dirichlet problem $(\nabla^2 + k^2)w = 0$, in D' with $w = g$ on S. Then there exists a constant $C > 0$, independent of w, such that*

$$|||w||| \le C\|g\|_{L^2(S)}\,, \tag{21}$$

where $|||\cdot||| := ||\cdot||_{H^m_{loc}(D')} + ||\cdot||_{L^2(D';(1+|x|)^{-\gamma})}$, $\gamma > 1$, $m > 0$ is an arbitrary integer, and H^m is the Sobolev space.

Proof. Let G be the Dirichlet Green's function of the Laplacian in D':

$$\left(\nabla^2 + k^2\right) G = -\delta(x-y) \text{ in } D', \quad G = 0 \text{ on } S, \tag{22}$$

$$\lim_{r\to\infty} \int_{|x|=r} \left| \frac{\partial G}{\partial |x|} - ikG \right|^2 ds = 0. \tag{23}$$

Let N be the unit normal to S pointing into D'. By Green's formula one has

$$w(x) = \int_S g(s) \frac{\partial G}{\partial N}(x,s)ds, \quad x \in D'\,. \tag{24}$$

The estimate for the $H^m_{loc}(D')$-norm part of (21) follows from this representation and from the Cauchy inequality:

$$|D^{(j)}w(x)| \le ||g||_{L^2(S)} \left\| \frac{\partial D_x^{(j)} G}{\partial N}(x,s) \right\| \le c(x)||g||_{L^2(S)},$$

where $c(x) \leq c(d)$ for all $x \in D'$ such that the distance $dist(x, S) \geq d > 0$.

For the L^2-weighted norm part of (21) let $R > 0$ be such that $D \subset B_R = \{x \in \mathbb{R}^3 \ : \ |x| < R\}$. Let $D'_R = B_R \setminus D$, and S_R be the boundary of B_R. Estimate

$$\left|\frac{\partial G}{\partial N}(x, s)\right| \leq \frac{c}{1 + |x|}, \quad |x| \geq R, \tag{25}$$

and (24) imply

$$\|w\|_{L^2(S_R)} \leq c\|g\|_{L^2(S)}, \tag{26}$$

where here and in the sequel c, C denote various constants. Also, using Cauchy-Schwarz inequality, (24), (25) and $\gamma > 1$, one gets

$$\|w\|_{L^2(|x|>R;(1+|x|)^{-\gamma})} \leq c\|g\|_{L^2(S)} \left\|\frac{1}{(1 + |x|)^{\gamma+1}}\right\|_{L^2(|x|>R)} \leq c\|g\|_{L^2(S)}. \tag{27}$$

To get the estimate for $\|w\|_{L^2(D'_R)}$ choose R such that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D'_R. Then ([5], p.189):

$$\|w\|_{H^m(D'_R)} \leq c[||(\Delta + k^2)w||_{\mathcal{H}^{m-2}(D'_R)} + ||w||_{H^{m-0.5}(S_R)} + ||w||_{H^{m-0.5}(S)}]. \tag{28}$$

The space $\mathcal{H}$ in the first term of the right-hand side in (28) is different from the usual Sobolev space, but this term is equal to zero anyway because $(\Delta + k^2)w = 0$.

Let $m = 0.5$ in (28). Then

$$\|w\|_{H^{0.5}(D'_R)} \leq c[||w||_{L^2(S_R)} + ||w||_{L^2(S)}]. \tag{29}$$

Since $w = g$ on S, then (26) and (29) imply

$$\|w\|_{L^2(D'_R)} \leq c||g||_{L^2(S)}. \tag{30}$$

$\square$

3 Numerical Experiments.

In this section we describe numerical results obtained by the MRC method with the optimal choice of sources for 2D and 3D obstacles. For a comparison, our earlier method (Multi-point MRC) that we used in [4] can be described as follows. Choose and fix J interior points $x_j, \ j = 1, 2, ..., J$ in D by an *ad hoc* method according to the geometry of the obstacle D. The discrepancy on the boundary S is minimized with respect to the coefficients $\mathbf{c}$. The number of points J was limited by the size of the resulting numerical minimization problem, so the accuracy of the scattering solution (i.e. the residual r^{min}) could not be made small for many obstacles. The method was not capable of treating complicated 3D obstacles. These limitations were removed by using the MRC method with optimal choice of sources. As we mentioned previously, [4] contains a favorable

comparison of the Multi-point MRC method with the BIEM (the Boundary Integral Equation Method).

Note that in a 2D case one has

$$\psi_\ell(x, z) = H_\ell^{(1)}(k|x - z|)e^{i\ell\theta},$$

where $(x - z)/|x - z| = e^{i\theta}$ instead of (4).

For a numerical implementation choose M nodes $\{t_m\}$ on the surface S of the obstacle D. Given $z \in D$ form N vectors

$$\mathbf{a}^{(n)} = \{\psi_\ell(t_m, z)\}_{m=1}^{M},$$

$n = 1, 2, \ldots, N$ of length M. Note that $N = 2L + 1$ for a 2D case, and $N = (L + 1)^2$ for a 3D case. It is convenient to normalize the norm in $\mathbb{R}^M$ by $1/M$, so

$$\|\mathbf{b}\|^2 = \frac{1}{M} \sum_{m=1}^{M} |b_m|^2, \quad \mathbf{b} = (b_1, b_2, ..., b_M).$$

Then $\|u_0\| = 1$.

Now let $\mathbf{b} = \{g_1(t_m)\}_{m=1}^{M}$, (see section 1), and minimize

$$\Phi(z, \mathbf{c}) = \|\mathbf{b} + A\mathbf{c}\|, \tag{31}$$

for $\mathbf{c} \in \mathbb{C}^N$, where A is the matrix containing vectors $\mathbf{a}^{(n)},\ n = 1, 2, \ldots, N$ as its columns. The same numerical procedure is also applied to subsequent iterative steps.

We used the Singular Value Decomposition (SVD) method (see e.g. [7]) to minimize (31). Small singular values $s_n < w_{min}$ of the matrix A are used to identify and delete linearly dependent or almost linearly dependent combinations of vectors $\mathbf{a}^{(n)}$. This spectral cut-off makes the minimization process stable, see the details in [4].

There is a variety of methods to minimize $\Phi(z, \mathbf{c}(z))$, since after the minimization in the coefficients $\mathbf{c}(z)$ it is just a 2D or 3D minimization in the region D. Our choice was Powell's method which imitates the conjugate gradients approach, but does not require analytical expressions for the gradient. The Brent method was used for a line minimization, see [7].

We conducted 2D numerical experiments for five obstacles: a circle, two ellipses of different eccentricity, a kite, and a triangle. The M=720 nodes t_m uniformly distributed on the interval $[0, 2\pi]$, were used to parameterize the boundary S. Each case was tested for wave numbers $k = 1.0$ and $k = 5.0$. Each obstacle was subjected to incident waves corresponding to $\alpha = (1.0, 0.0)$ and $\alpha = (0.0, 1.0)$. The results are shown in Table I. The column N_{iter} shows the number of iterations (number of source points) at the end of the iterative process. The process was stopped after the algorithm reached the sought tolerance $\epsilon = 0.002$, or $N_{max} = 100$. Values $L = 5$ and $M = 720$ were used in all 2D experiments.

The last column $(MRC - BIEM)/BIEM$ shows the discrepancy in the scattering amplitude computed by the MRC and BIEM methods. The values

shown are the L_2 norms of the difference of the scattering amplitude obtained by MRC and BIEM, over the L_2 norm of the scattering amplitude obtained by BIEM. We followed [3] for the BIEM implementation using 64 points on the boundary S in every 2D experiment. No comparison is provided for a triangular obstacle, since it requires a complete rewriting of the BIEM code to accommodate the corner points. No such rewriting is required for the MRC method. Table I shows that for the value of tolerance $\epsilon = 0.002$ the computed scattering amplitude is in an excellent agreement with the scattering amplitude computed using BIEM.

Concerning the efficiency of the methods, for simple geometries the Multi-point MRC (see [4]) is the fastest, provided that the required accuracy can be achieved by a relatively small number J of the interior points (sources) used simultaneously. This assures the resulting matrices being of a manageable size. Otherwise, one has to use the proposed MRC with the optimal choice of sources, which takes a significantly longer time to run, but can accomplish the solution of scattering problems untractable by single step methods, such as the Multi-point MRC or BIEM.

In another modification of the MRC method (Random point MRC) sources were placed randomly inside the obstacle D, and after each such placement the discrepancy on the boundary was minimized. Unlike (5) the discrepancy was not additionally minimized to obtain an optimal source placement. While this method also achieved a better fit than the original MRC described in [4], the optimally placed MRC method described here achieves an order of magnitude improvement in run time over the Random point MRC.

Experiment 2D-I. The boundary S is the circle of radius 1.0 centered in the origin.

Experiment 2D-II. The boundary S is the ellipse described by

$$\mathbf{r}(t) = (2.0\cos t,\ \sin t), \quad 0 \le t < 2\pi\,. \tag{32}$$

Experiment 2D-III. The boundary S is the ellipse described by

$$\mathbf{r}(t) = (0.1\cos t,\ \sin t), \quad 0 \le t < 2\pi\,. \tag{33}$$

Experiment 2D-IV. The kite-shaped boundary S (see [3], Section 3.5) is described by

$$\mathbf{r}(t) = (-0.65 + \cos t + 0.65\cos 2t,\ 1.5\sin t), \quad 0 \le t < 2\pi\,. \tag{34}$$

Experiment 2D-V. The boundary S is the triangle with vertices $(-1.0, 0.0)$ and $(1.0, \pm 1.0)$.

The 3D numerical experiments were conducted for 4 obstacles: a sphere, two ellipsoids, and a cube. We used $L = 5$, $\epsilon = 0.002$, $w_{min} = 10^{-12}$. The number M of the fit points on the boundary S is indicated in the description of the obstacles. The scattered field for each obstacle was computed for two incident directions $\alpha_i,\ i = 1, 2$. The first unit vector α_1 is denoted by (1) in Table II, $\alpha_1 = (1.0, 0.0, 0.0)$. The second one is denoted by (2), $\alpha_2 = (0.0, 1/\sqrt{2}, 1/\sqrt{2})$.

Experiment 3D-I. The boundary S is the sphere of radius 1, with $M = 450$.

Experiment 3D-II. The boundary S is the surface of the ellipsoid $x^2/4 + y^2 + z^2 = 1$ with $M = 450$.

Experiment 3D-III. The boundary S is the surface of the ellipsoid $x^2/16 + y^2 + z^2 = 1$ with $M = 450$.

Experiment 3D-IV. The boundary S is the surface of the cube $[-1, 1]^3$ with $M = 1350$.

4 Conclusions.

For a 2D or 3D obstacle Rayleigh conjectured that the acoustic field u in the exterior of the obstacle is given by

$$u(x, \alpha) = e^{ik\alpha \cdot x} + \sum_{\ell=0}^{\infty} A_\ell(\alpha)\psi_\ell, \quad \psi_\ell := Y_\ell(\alpha')h_\ell(kr), \quad \alpha' = \frac{x}{r}. \tag{35}$$

While this conjecture (RC) is false for many obstacles, it has been modified to obtain a valid representation for the solution of (1)-(2). This representation (Theorem 1) is called the Modified Rayleigh Conjecture (MRC), and is, in fact, not a conjecture, but a Theorem.

Can one use this approach to obtain solutions to various scattering problems? A numerical implementation of the MRC with just one source point as presented in section 2 is not successful, since it requires a large value of L to obtain an acceptable accuracy. However, for such large values of L the involved Hankel functions exceed lower order Hankel functions by many orders of magnitude, which, in turn, requires an unacceptably large precision of computations to account for delicate cancellations. On the other hand, as we show here and in [4], the MRC can be efficiently implemented to obtain accurate numerical solutions of obstacle scattering problems if one uses multiple source points.

The algorithm presented in this paper, MRC with optimal choice of sources, was successfully applied to various 2D and 3D obstacle scattering problems. This algorithm is a significant improvement over previous MRC implementation described in [4]. The improvement is achieved by allowing the required minimizations to be done iteratively, while the previous methods were limited by the problem size constraints. Still, the Multi-point MRC presented in [4], remains very efficient for problems of a relatively simple geometry, and it was favorably compared to the Boundary Integral Equation Method.

The MRC methods have an additional attractive feature that they can easily treat obstacles with complicated geometry (e.g. obstacles with edges and corners). Unlike the BIEM, it is easily modified to treat different obstacle shapes.

Further research on MRC algorithms is conducted. It is hoped that the MRC in its various implementation can emerge as a valuable and efficient alternative to more established methods.

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Table I: Normalized residuals attained in the numerical experiments for 2D obstacles, $\|\mathbf{u_0}\| = 1$.

Experiment	k	α	N_{iter}	r^{min}	(MRC-BIEM)/BIEM
I	1.0	$(1.0, 0.0)$	1	0.0000	0.0001
Circle	5.0	$(1.0, 0.0)$	21	0.0020	0.0001
II	1.0	$(1.0, 0.0)$	20	0.0010	0.0001
Ellipse	1.0	$(0.0, 1.0)$	20	0.0018	0.0001
	5.0	$(1.0, 0.0)$	53	0.0010	0.0001
	5.0	$(0.0, 1.0)$	45	0.0020	0.0001
III	1.0	$(1.0, 0.0)$	100	0.0041	0.0008
Ellipse	1.0	$(0.0, 1.0)$	100	0.0027	0.0000
	5.0	$(1.0, 0.0)$	100	0.0058	0.0004
	5.0	$(0.0, 1.0)$	100	0.0037	0.0012
IV	1.0	$(1.0, 0.0)$	53	0.0020	0.0001
Kite	1.0	$(0.0, 1.0)$	32	0.0020	0.0001
	5.0	$(1.0, 0.0)$	75	0.0020	0.0003
	5.0	$(0.0, 1.0)$	68	0.0020	0.0001
V	1.0	$(1.0, 0.0)$	55	0.0020	
Triangle	1.0	$(0.0, 1.0)$	48	0.0017	
	5.0	$(1.0, 0.0)$	72	0.0019	
	5.0	$(0.0, 1.0)$	80	0.0020	

Table II: Normalized residuals attained in the numerical experiments for 3D obstacles, $\|\mathbf{u_0}\| = 1$.

Experiment	k	α_i	N_{iter}	r^{min}
I	1.0		1	0.0000
Sphere	5.0		43	0.0019
II	1.0	(1)	20	0.0019
Ellipsoid	1.0	(2)	25	0.0019
	5.0	(1)	44	0.0019
	5.0	(2)	58	0.0019
III	1.0	(1)	12	0.0016
Ellipsoid	1.0	(2)	35	0.0020
	5.0	(1)	55	0.0020
	5.0	(2)	67	0.0020
IV	1.0	(1)	12	0.0019
Cube	1.0	(2)	7	0.0019
	5.0	(1)	70	0.0019
	5.0	(2)	35	0.0020

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On Kurzweil-Henstock type integrals with respect to abstract derivation bases for Riesz-space-valued functions

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Abstract

Some kinds of integral, like variational, Kurzweil-Henstock and (SL)-integral are introduced and investigated in the context of Riesz-space-valued functions and with respect to abstract derivation bases. Some versions of the Fundamental Theorem of Integral Calculus are given.

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Key words: Riesz spaces, derivation basis, Kurzweil-Henstock integral, Variational integral, Lusin property, interval functions, Fundamental Formula of Calculus.

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1 Introduction

In this note we consider three kinds of definitions of Kurzweil-Henstock type integrals with respect to an abstract derivation bases: the original definition of the Kurzweil-Henstock integral based on generalized Riemann sums, variational integral and the so-called SL-integral, introduced in [10]. The relation between those integrals depends on whether we consider them in application to the real-valued functions or to the Banach-space-valued functions or at last to the functions with values in Riesz spaces (vector lattices). In the case of real-valued functions all the three integrals are equivalent (see [4] and [10]). In the Banach case the SL-integral is equivalent to the variational integral, but both are strictly included into the Kurzweil-Henstock integral (see [19]). As for the Riesz-space-valued case, we shall check here that the Kurzweil-Henstock integral is equivalent to the variational integral, and the SL-integral is not equivalent to them. All three integrals are equivalent only if we impose some additional assumption on the involved Riesz space.

These subjects are investigated in Section 4. In Section 2 we introduce some definitions and notations related to the notion of the abstract derivation basis and to Riesz spaces. In Section 3 we introduce a Kurzweil-Henstock type integral in our setting and study some of its properties. In Section 5 we prove some versions of the Fundamental Formula of Integral Calculus in our abstract setting.

2 Preliminaries

We introduce some definitions and notations. A *derivation basis* (or simply a *basis*) $\mathcal{B}$ in a measure space $(X, \mathcal{M}, \mu)$ is a filter base on the product space $\mathcal{I} \times X$, where $\mathcal{I}$ is a family of measurable subsets of X having positive measure μ and called *generalized intervals* or *$\mathcal{B}$-intervals.* That is $\mathcal{B}$ is a nonempty collection of subsets of $\mathcal{I} \times X$ so that each $\beta \in \mathcal{B}$ is a set of pairs (I, x), where $I \in \mathcal{I}$, $x \in X$, and $\mathcal{B}$ has the *filter base property:* $\emptyset \notin \mathcal{B}$ and for every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta \in \mathcal{B}$ such that $\beta \subset \beta_1 \cap \beta_2$. So each basis is an ordered directed set and the order is given by the "reversed" inclusion. We shall refer to the elements β of $\mathcal{B}$ as *basis sets.*

In this paper we shall always suppose that $(I, x) \in \beta$ implies $x \in I$, although it is not the case in the general theory (see [15], [18]). For a set $E \subset X$ and $\beta \in \mathcal{B}$ we write

$$\beta(E) = \{(I, x) \in \beta : I \subset E\}, \quad \beta[E] = \{(I, x) \in \beta : x \in E\}.$$

We shall assume that for any two basis sets $\beta_1, \beta_2 \in \mathcal{B}$ and for any disjoint sets E_1, E_2 such that

$E_1 \cup E_2 = X$, the set $\beta_1[E_1] \cup \beta_2[E_2]$ is a basis set in $\mathcal{B}$.

In the case of a topological space X we say that a basis $\mathcal{B}$ is a *Vitali basis*, if for any x, for each neighborhood $U(x)$ of x and for every $\beta \in \mathcal{B}$ the set $\{(I, x) \in \beta, I \subset U(x)\}$ is nonempty. The simplest Vitali derivation basis in $\mathbb{R}^m$ is the *full interval basis*. In this case, $\mathcal{I}$ is the set of all m-dimensional intervals in $\mathbb{R}^m$ and each basis set is defined by a positive function δ on $\mathbb{R}^m$ called *gage* as

$$\beta_\delta = \{(I, x) : I \in \mathcal{I},\ x \in I \subset U(x, \delta(x))\},$$

where $U(x, \delta(x))$ is the ball of center x and radius $\delta(x)$. So the full interval basis is the family $(\beta_\delta)_\delta$ where δ runs over the set of all possible gages. Some other examples of bases in $\mathbb{R}^m$, also defined by gages, are considered in [15]. Among them are so called regular bases which are defined with $\mathcal{I}$ being a set of intervals with their regularity restricted from below by some positive number (the regularity of an interval is the ratio of its Lebesgue measure to that of the smallest cube containing it). The *dyadic basis* (see [3]) is defined as an interval basis determined by a class of dyadic intervals, i. e. intervals which are Cartesian product of intervals of the form $\left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]$. Different kind of symmetrical bases for which $(I, x) \in \beta$ implies that I is centered at x, are widely used in harmonic analysis (see [23]). Note that the notion of a derivation basis which is used in Henstock theory is slightly different from the one in [5] where it is defined locally, for each point $x \in X$. All the above interval bases can be considered not only in the usual Lebesgue measure space $(\mathbb{R}^m, \mathcal{M}, \lambda_m)$, with λ_m being the Lebesgue m-dimensional measure but also in measure spaces generated by different kind of Stieltjes measures μ on $\mathcal{I}$ leading to the respective Henstock-Stieltjes integrals. An interesting example of a basis can be considered in the Wiener space with $\mathcal{I}$ being the set of all cylindrical intervals in it (see [14]). Finally, we mention a measure space $(X, \mathcal{M}, \mu)$ with X being a locally compact Abelian zero-dimensional group with Haar measure μ in it and with $\mathcal{I}$ being formed by cosets of all subgroups of this group (see [20]).

A finite collection $\pi \subset \beta$ is called a *β-partition* if, for any distinct elements (I', x') and (I'', x'') in π, the $\mathcal{B}$-intervals I' and I'' are non-overlapping (i.e. their intersection is a set of measure μ zero). If a partition $\pi = \{(I_i, x_i)\} \subset \beta(I)$ for some $I \in \mathcal{I}$ is such that $\cup_i I_i = I$, then we say that π is a *β-partition of I*. We denote by the symbol $\Pi(\beta; I)$ the totality of all β-partitions of a generic $\mathcal{B}$-interval I.

We say that a basis $\mathcal{B}$ has the *partitioning property* if the following conditions hold: (i) for each finite collection $I_0, I_1, \ldots, I_n$ of $\mathcal{B}$-intervals with $I_1, \ldots, I_n \subset I_0$ the difference $I_0 \setminus \cup_{i=1}^n I_i$ can be expressed as a finite union of pairwise non-overlapping $\mathcal{B}$-intervals; (ii) for each $\mathcal{B}$-interval I and for any $\beta \in \mathcal{B}$

there exists $\pi \in \Pi(\beta; I)$. In the particular case of the full interval basis on $I\!R$, this property has long been known as the Cousin lemma. For the full interval basis in $I\!R^m$, the partitioning property can also be established without difficulty. But for some bases this property was proved only recently (see [6]), and there are bases for which it is not valid at all or holds true only in some weaker sense as it is in the case of the symmetric approximate basis (see [17]).

We say that a basis $\mathcal{B}$ *ignores* a set $E \subset X$ if there exists a basis set $\beta \in \mathcal{B}$ such that $\beta[E]$ is empty. We say that $\mathcal{B}$ is a *complete* basis if it ignores no point. If $(X, \mathcal{M}, \mu)$ is a measure space, we say that $\mathcal{B}$ is an *almost complete* basis if it ignores n o set of non-zero measure μ. Having a complete basis $\mathcal{B}$ on E, $\mu(E) > 0$, we can always extend it to an almost complete basis by including into $\mathcal{B}$ together with each basis set β all the sets of the form $\beta \setminus \beta[K]$, where K is any set of measure μ zero. We shall denote this almost complete extension of the basis $\mathcal{B}$ by $\mathcal{B}_0$. (Filter base property for the basis $\mathcal{B}_0$ follows easily from the fact that $\mathcal{B}$ has this property, from assumption $\mu(E) > 0$ and from obvious properties of the family of the sets of measure zero.) Now we consider a Dedekind complete Riesz space R. We add to R two extra elements, $+\infty$ and $-\infty$, extending ordering and operations, in such a way that

$$\begin{cases} +\infty \geq r, \ \forall\, r \in R, \\ 0 \cdot (+\infty) = 0, \\ \lambda \cdot (+\infty) = +\infty, \ \forall\, \lambda \in I\!R^+, \\ +\infty + r = +\infty, \ \forall\, r \in R \cup \{+\infty\}, \end{cases}$$

and

$$\begin{cases} -\infty \leq r, \ \forall\, r \in R, \\ 0 \cdot (-\infty) = 0, \\ \lambda \cdot (-\infty) = -\infty, \forall\, \lambda \in I\!R^+, \\ -\infty + r = -\infty, \ \forall\, r \in R \cup \{-\infty\} \\ \lambda \cdot (+\infty) = -\infty, \ \lambda \cdot (-\infty) = +\infty, \forall\, \lambda \in I\!R^-. \end{cases}$$

Let $\overline{R} = R \bigcup \{+\infty, -\infty\}$. A nonempty set $T \subset R$ is said to be *upper bounded* if there exists $s_1 \in R$ such that $s_1 \geq t$ for all $t \in T$, *lower bounded* if there exists $s_2 \in R$ such that $s_2 \leq t$ for all $t \in T$, *bounded* if it is both upper and lower bounded. By convention, we will say that the supremum of any not upper bounded nonempty subset of R is $+\infty$ and the infimum of any not lower bounded nonempty subset of R is $-\infty$.

Given a sequence $(r_n)_n$ in $\overline{R}$, we define

$$\limsup_n r_n = \inf_n [\sup_{m \geq n} r_m],$$

$$\liminf_n r_n = \sup_n [\inf_{m \geq n} r_m].$$

Given a net $(r_\eta)_{\eta \in \Lambda}$ in $\overline{R}$, where $(\Lambda, \geq) \neq \emptyset$ is a directed set, let

$$\limsup_\eta r_\eta = \inf_\eta [\sup_{\zeta \geq \eta} r_\zeta],$$

$$\liminf_\eta r_\eta = \sup_\eta [\inf_{\zeta \geq \eta} r_\zeta].$$

We say that $(r_\eta)_\eta$ *order converges* (or in short (o)-*converges*) to $r \in R$ if $r = \limsup_\eta r_\eta = \liminf_\eta r_\eta$, and we write $(o)\lim_{\eta \in \Lambda} r_\eta = r$. An (o)-*net* $(r_\eta)_{\eta \in \Lambda}$ is a monotone decreasing net of elements of $\overline{R}$, such that $\inf_{\eta \in \Lambda} r_\eta = 0$. In particular this defines also the notion of (o)-sequence.

A Riesz space R *satisfies property* σ if, given any sequence $(u_n)_n$ in R with $u_n \geq 0 \; \forall \, n \in I\!N$, there exists a sequence $(\lambda_n)_n$ of positive real numbers, such that the sequence $(\lambda_n u_n)_n$ is bounded in R. A Riesz space R *satisfies the Swartz property* if there exists a sequence $(h_n)_n$ in R such that, for each $x \in R$, $\exists \, k, n \in I\!N$ such that $|x| \leq k \, h_n$ (see [21]).

3 Differential and Integral Calculus

We now introduce a concept of "generalized derivative" (with respect to a basis) for $\mathcal{B}$-interval functions with values in Riesz spaces. From now on in this and in the next section, we fix a measure space $(X, \mathcal{M}, \mu)$ and a complete basis $\mathcal{B}$ in it. We shall always suppose that all the suprema and the sums "along the empty set" are equal to zero.

Let R be a Dedekind complete Riesz space and let a $\mathcal{B}$-interval function $\tau : \mathcal{I} \to R$ be given. We say that τ is *additive* if $\tau(I' \bigcup I'') = \tau(I') + \tau(I'')$ whenever I' and I'' are any two non-overlapping $\mathcal{B}$-intervals. The function τ is said to be (o)-*continuous* (or simply *continuous*) *at the (fixed) point* $x_0 \in X$ if

$$\inf_{\beta \in \mathcal{B}} [\sup \{|\tau(I)| : (I, x_0) \in \beta\}] = 0. \tag{1}$$

Given $\emptyset \neq E \subset X$, we say that an additive function τ is (o)-*continuous* (or *continuous*) *in* E if it is continuous at every point $x_0 \in E$. In particular, τ is said to be (o)-*continuous* (or *continuous*) if it is (o)-continuous (or continuous) in X. The function τ is (u)-*continuous in* E if

$$\inf_{\beta \in \mathcal{B}} [\sup \{|\tau(I)| : (I, x) \in \beta[E]\}] = 0. \tag{2}$$

We say that an additive function τ is (u)-*differentiable in* E if there exists a function $g : E \to R$ such that

$$\inf_{\beta \in \mathcal{B}} \left[\sup \left\{ \left| \frac{\tau(I)}{\mu(I)} - g(x) \right| : (I, x) \in \beta[E] \right\} \right] = 0, \tag{3}$$

or, equivalently, if there exist a function $g : E \to R$ and an (o)-net $(p_\beta)_{\beta \in \mathcal{B}}$ such that, for all $\beta \in \mathcal{B}$ and for every $(I, x) \in \beta[E]$, we get:

$$|\tau(I) - \mu(I)\, g(x)| \le \mu(I)\, p_\beta.$$

(Of course, here it is implicit the convention that, if $\beta[E]$ is empty for some element $\beta \in \mathcal{B}$, then every $\mathcal{B}$-interval function τ is (u)-differentiable in E: this can be viewed also as an immediate consequence of our previous conventions).

The function g in (3) is called the (u)-*derivative* (or simply *derivative*) *of* τ *in* E. More simply, we say that the function τ is (u)-*continuous* (resp. (u)-*differentiable*) if it is (u)-continuous in X (resp. (u)-differentiable in X).

Furthermore, the function τ is said to be (o)-*differentiable in* E if it is (u)-differentiable in $\{x\}$ for all $x \in E$. We note that, in the case of the bases defined by a constant gage δ (i.e. $\beta_\delta = \{(I, x) : x \in I \subset U(x, \delta)\}$, we get uniform continuity and differentiability. Moreover, it is easy to check that, if $R = I\!R$ and we have the usual interval bases, then (o)- and (u)-differentiability at a point coincide with usual differentiability.

The *upper derivative* of a $\mathcal{B}$-interval function τ at a point x in E is defined as

$$h'(x) = \inf_{\beta \in \mathcal{B}} \left[\sup \left\{ \frac{\tau(I)}{\mu(I)} : (I, x) \in \beta \right\} \right]. \tag{4}$$

Similarly, the *lower derivative* is defined as

$$g'(x) = \sup_{\beta \in \mathcal{B}} \left[\inf \left\{ \frac{\tau(I)}{\mu(I)} : (I, x) \in \beta \right\} \right]. \tag{5}$$

If we apply these definitions to the particular bases described in the previous section we get the respective known derivatives. In particular the regular interval basis defines the *regular derivative,* the dyadic basis defines the *dyadic derivative* and so on.

We now prove the following:

Theorem 3.1 *Let* τ *be an* R*-valued* $\mathcal{B}$*-interval function and* $\emptyset \neq E \subset X$*. Then the following conditions are equivalent:*

i) τ *is* (u)*-differentiable in* E*;*

ii) $\inf_{\beta\in\mathcal{B}}\ \rho_E(\beta)=0$, *where*

$$\rho_E(\beta)=\sup\left\{\left|\frac{\tau(I')}{\mu(I')}-\frac{\tau(I'')}{\mu(I'')}\right| : (I'x),(I'',x)\in\beta[E]\right\}.$$

In this case, $g'(x)=h'(x)\ \forall\, x\in E$, *where* g' *and* h' *are defined by (4) and (5) respectively. Moreover, the* (u)*-derivative coincides with* g' *and* h'*, and is uniquely determined.*

Proof: The implication i) $\Longrightarrow$ ii) and uniqueness of (u)-derivative are straightforward. Now we prove ii) $\Longrightarrow$ i). For every $\beta\in\mathcal{B}$, let

$$\alpha(\beta)=\sup\left\{\frac{\tau(I)}{\mu(I)} : (I,x)\in\beta[E]\right\},\quad \kappa(\beta)=\inf\left\{\frac{\tau(I)}{\mu(I)} : (I,x)\in\beta[E]\right\},$$
$$\rho(\beta)=\sup\left\{\left|\frac{\tau(I')}{\mu(I')}-\frac{\tau(I'')}{\mu(I'')}\right| : (I',x),(I'',x)\in\beta[E]\right\}.$$

We have:

$$\frac{\tau(I')}{\mu(I')}\le\rho(\beta)+\frac{\tau(I'')}{\mu(I'')}\quad\forall\,(I',x),(I'',x)\in\beta[E].$$

Taking the suprema and the infima, we get

$$\alpha(\beta)\le\rho(\beta)+\kappa(\beta). \tag{6}$$

Moreover,

$$\left|\frac{\tau(I')}{mu(I')}-\frac{\tau(I'')}{\mu(I'')}\right|\le\alpha(\beta)-\kappa(\beta)\quad\forall\,(I',x),(I'',x)\in\beta[E];$$

taking the supremum we obtain

$$\rho(\beta)\le\alpha(\beta)-\kappa(\beta),$$

and hence we have

$$\rho(\beta)=\alpha(\beta)-\kappa(\beta).$$

We observe that the nets $(\alpha(\beta))_{\beta\in\mathcal{B}}$ and $(\kappa(\beta))_{\beta\in\mathcal{B}}$ are monotone decreasing and increasing respectively, and $(\rho(\beta))_{\beta\in\mathcal{B}}$ is monotone decreasing, and thus, thanks also to (6), there exist *in* R the following (o)-limits:

$$(o)\lim_{\beta\in\mathcal{B}}\ \alpha(\beta)=\inf_{\beta\in\mathcal{B}}\ \alpha(\beta),(o)\lim_{\beta\in\mathcal{B}}\kappa(\beta)=\sup_{\beta\in\mathcal{B}}\ \kappa(\beta),(o)\lim_{\beta\in\mathcal{B}}\ \rho(\beta)=\inf_{\beta\in\mathcal{B}}\ \rho(\beta)$$

(indeed, it is enough to observe that there exists at least an element $\beta_0\in\mathcal{B}$ such that $\alpha(\beta)\in R\ \forall\,\beta\in\mathcal{B}$, $\beta\subset\beta_0$, since by hypothesis

$$(o)\lim_{\beta\in\mathcal{B}}\ \rho(\beta)=\inf_{\beta\in\mathcal{B}}\ \rho(\beta)=0). \tag{7}$$

From (7) we obtain:

$$0 = (o) \lim_{\beta \in \mathcal{B}} \rho(\beta) = (o) \lim_{\beta \in \mathcal{B}} \alpha(\beta) - (o) \lim_{\beta \in \mathcal{B}} \kappa(\beta).$$

Thus there exists an (o)-net $(w_\beta)_{\beta \in \mathcal{B}}$ such that

$$0 \leq \alpha(\beta) - \kappa(\beta) \leq w_\beta \quad \forall \beta \in \mathcal{B}, \beta \subset \beta_0,$$

and hence

$$\frac{\tau(I)}{\mu(I)} \leq w_\beta + \kappa(\beta)$$

whenever $(I, x) \in \beta[E]$, that is whenever $x \in E$ and $(I, x) \in \beta$. Thus it follows that, whenever $(I, x) \in \beta[E]$, we have:

$$h'(x) \leq w_\beta + \kappa(\beta) \leq w_\beta + \frac{\tau(I)}{\mu(I)},$$

and so

$$\frac{\tau(I)}{\mu(I)} - g'(x) \geq \frac{\tau(I)}{\mu(I)} - h'(x) \geq -w_\beta. \tag{8}$$

Analogously, it is possible to check that, whenever $(I, x) \in \beta[E]$, we get:

$$g'(x) \geq -w_\beta + \alpha(\beta) \geq -w_\beta + \frac{\tau(I)}{\mu(I)},$$

and so

$$\frac{\tau(I)}{\mu(I)} - h'(x) \leq \frac{\tau(I)}{\mu(I)} - g'(x) \leq w_\beta. \tag{9}$$

From (8) and (9) it follows that

$$\left| \frac{\tau(I)}{\mu(I)} - g'(x) \right| \leq w_\beta, \quad \left| \frac{\tau(I)}{\mu(I)} - h'(x) \right| \leq w_\beta \tag{10}$$

whenever $x \in E$ and $(I, x) \in \beta[E]$. From (10) it follows that

$$0 \leq \sup \left\{ \left| \frac{\tau(I)}{\mu(I)} - g'(x) \right| : (I, x) \in \beta[E] \right\} \leq w_\beta \text{ and}$$

$$0 \leq \sup \left\{ \left| \frac{\tau(I)}{\mu(I)} - h'(x) \right| : (I, x) \in \beta[E] \right\} \leq w_\beta$$

for any $\beta \in \mathcal{B}$, $\beta \subset \beta_0$. Thus ii) implies i). From (10) it follows that $g'(x) = h'(x) \; \forall x \in E$ and that $g' = h'$ is the requested (u)-derivative. This concludes the proof. $\square$

We now introduce a Kurzweil-Henstock type integral with respect to an abstract derivation basis (for the real case, see [4], [15]). If E is a fixed $\mathcal{B}$-interval, $f : E \to R$ and $\pi = \{(J_i, \xi_i) : i = 1, \ldots, q\}$ is a partition of E, we will call *Riemann sum* associated with π and we will write it by the symbol $S(f, \pi)$ the quantity $\displaystyle\sum_{i=1}^{q} \mu(J_i) \, f(\xi_i)$.

Definition 3.2 Let R be a Dedekind complete Riesz space, $\mathcal{B}$ be a fixed complete basis having the partitioning property and $E \subset X$ be a $\mathcal{B}$-interval. We say that $f : E \to R$ is *Kurzweil-Henstock integrable* (in brief, *$H_{\mathcal{B}}$-integrable*) *on E* (with respect to $\mathcal{B}$) if there exists an element $Y \in R$ such that

$$\inf_{\beta \in \mathcal{B}} \left(\sup \left\{|S(f, \pi) - Y| : \pi \in \Pi(\beta; E)\right\}\right) = 0. \tag{11}$$

In this case we write $(H_{\mathcal{B}}) \displaystyle\int_E f = Y$.

It is easy to see that the element Y in (11) is uniquely determined.

Definition 3.3 Let R be a Dedekind complete Riesz space, $\mathcal{B}$ be a fixed complete basis having the partitioning property and $E \subset X$ be a $\mathcal{B}$-interval. We say that a family $(f_\lambda)_{\lambda \in \Lambda}$ of $H_{\mathcal{B}}$-integrable functions $f_\lambda : E \to R$ is *uniformly $H_{\mathcal{B}}$-integrable on E* if there exists an (o)-net$(p_\beta)_{\beta \in \mathcal{B}}$ such that

$$\sup \left\{ \left| S(f_\lambda, \pi) - (H_{\mathcal{B}}) \int_E f_\lambda \right| : \pi \in \Pi(\beta; E) \right\} \leq p_\beta \tag{12}$$

for all $\lambda \in \Lambda$.

The Cauchy criterion for $H_{\mathcal{B}}$-integrability can be proved repeating the argument in [3] where a less general class of bases is considered. In a similar way the following Cauchy criterion for the uniform $H_{\mathcal{B}}$-integrability can be established.

Theorem 3.4 *Let $R, \mathcal{B}$ and E be as in Definition 3.3. A family $(f_\lambda)_{\lambda \in \Lambda}$ of functions $f_\lambda : E \to R$ is uniformly $H_{\mathcal{B}}$-integrable on E if and only if there exists an (o)-net $(p_\beta)_{\beta \in \mathcal{B}}$ such that*

$$\sup \left\{|S(f_\lambda, \pi_1) - S(f_\lambda, \pi_2)| : \pi_1, \pi_2 \in \Pi(\beta; E)\right\} \leq p_\beta$$

for all $\lambda \in \Lambda$.

The following two propositions can be proved repeating the arguments of the corresponding propositions in [3].

Proposition 3.5 *If $E = I \cup J$ where E, I, J are $\mathcal{B}$-intervals, I and J are non-overlapping and f is $H_{\mathcal{B}}$-integrable on I and on J, then f is $H_{\mathcal{B}}$-integrable on E and $(H_{\mathcal{B}}) \displaystyle\int_E f = (H_{\mathcal{B}}) \int_I f + (H_{\mathcal{B}}) \int_J f$.*

Proposition 3.6 *If f is $H_{\mathcal{B}}$-integrable on a $\mathcal{B}$-interval I and $J \subset I$ is a $\mathcal{B}$-interval, then f is $H_{\mathcal{B}}$-integrable on J too.*

Uniform versions of the above two propositions can also be established, the second one being a consequence of Theorem 3.4 in the same way as Proposition 3.6 is a consequence of the usual Cauchy criterion.

Proposition 3.7 *If $E = I \cup J$ where E, I, J are $\mathcal{B}$-intervals, I and J are non-overlapping and a family $(f_\lambda)_{\lambda\in\Lambda}$ of functions $f_\lambda : E \to R$ is uniformly $H_{\mathcal{B}}$-integrable on I and on J, then the family $(f_\lambda)_{\lambda\in\Lambda}$ is uniformly $H_{\mathcal{B}}$-integrable on E.*

Proposition 3.8 *If a family $(f_\lambda)_{\lambda\in\Lambda}$ of functions $f_\lambda : E \to R$ is uniformly $H_{\mathcal{B}}$-integrable on a $\mathcal{B}$-interval I and $J \subset I$ is a $\mathcal{B}$-interval, then the family $(f_\lambda)_{\lambda\in\Lambda}$ is uniformly $H_{\mathcal{B}}$-integrable on J too.*

It follows from Propositions 3.5 and 3.6 that for any $H_{\mathcal{B}}$-integrable function $f : E \to R$ the indefinite $H_{\mathcal{B}}$-integral is defined as an additive R-valued $\mathcal{B}$-interval function on the family of all $\mathcal{B}$-intervals in E. We shall denote it by

$$F(I) = (H_{\mathcal{B}}) \int_I f. \tag{13}$$

We now prove the following version of the Saks-Henstock Lemma (a little bit less general version is proved in [3]; see also [12], Lemma 12, pp. 353-354):

Lemma 3.9 *If f is $H_{\mathcal{B}}$-integrable on E and F is as in (13), then*

$$\inf_{\beta\in\mathcal{B}} \left[\sup \left\{ \sum_{(I,x)\in\pi} |\, \mu(I)\, f(x) - F(I)| : \pi \in \Pi(\beta; E) \right\} \right] = 0. \tag{14}$$

Proof: By (11), there exists an (o)-net $(p_\beta)_{\beta\in\mathcal{B}}$ such that

$$\sup \left\{ \left| \sum_{(I,x)\in\pi} \mu(I)\, f(x) - (H_{\mathcal{B}}) \int_E f \right| : \pi \in \Pi(\beta; E) \right\} \le p_\beta \tag{15}$$

for every $\beta \in \mathcal{B}$. Let $\beta \in \mathcal{B}$ and $\{(J_i, \xi_i), i = 1, \dots, q\} = \pi \in \Pi(\beta; E)$. By Proposition 3.6, f is integrable on J_i, $i = 1, \dots, q$. Thus for each i there exists an (o)-net $(p_{\beta_i})_{\beta_i\in\mathcal{B}(J_i)}$ such that

$$\sup \left\{ \left| \sum_{(I,x)\in\pi} \mu(I)\, f(x) - (H_{\mathcal{B}}) \int_{J_i} f \right| : \pi \in \Pi(\beta_i; J_i) \right\} \le p_{\beta_i}. \tag{16}$$

Now, fix arbitrarily $\emptyset \ne L \subset \{1, \dots, q\}$. Let $\pi_i \in \Pi(\beta_i; J_i)$, and set

$$\pi_0 = \{(J_i, \xi_i) \in \pi : i \in L\} \bigcup (\bigcup_{i\notin L} \pi_i).$$

We can suppose that $\beta_i \in \beta[J_i]$. Then $\pi_0 \in \Pi(\beta; E)$ and hence

$$\left| S(f, \pi_0) - (H_B) \int_E f \right| \le p_\beta.$$

Thus we have

$$\begin{aligned}
0 &\le \left|\sum_{i\in L} \mu(J_i)\, f(\xi_i) - \sum_{i\in L} (H_{\mathcal{B}}) \int_{J_i} f\right| \\
&= \left|S(f,\pi_0) - (H_{\mathcal{B}}) \int_E f + \sum_{i\notin L} (H_{\mathcal{B}}) \int_{J_i} f - \sum_{i\notin L} S(f,\pi_i)\right| \\
&\le \left|S(f,\pi_0) - (H_{\mathcal{B}}) \int_E f\right| + \left|\sum_{i\notin L} (H_{\mathcal{B}}) \int_{J_i} f - \sum_{i\notin L} S(f,\pi_i)\right| \\
&\le \left|S(f,\pi_0) - (H_{\mathcal{B}}) \int_E f\right| + \sum_{i=1}^{n} \left|(H_{\mathcal{B}}) \int_{J_i} f - S(f,\pi_i)\right| \\
&\le p_\beta + \sum_{i=1}^{n} p_{\beta_i}.
\end{aligned}$$

Considering this inequality for a fixed β and for arbitrary β_i's we can take the (o)-limit to get

$$0 \le \left|\sum_{i\in L} \mu(J_i)\, f(\xi_i) - \sum_{i\in L} (H_{\mathcal{B}}) \int_{J_i} f\right| \le p_\beta. \tag{17}$$

We now observe that, since R is a Dedekind complete Riesz space, by virtue of the Maeda-Ogasawara-Vulikh representation theorem (see [1]) there exists a compact extremely disconnected topological space Ω, such that R can be embedded Riesz isomorphically as a solid subset of $C_\infty(\Omega) = \{f : \Omega \to \widetilde{I\!R} :$ f is continuous, and the set $\{\omega \in \Omega : |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. From (17), for all $\omega \in \Omega$ and for each $\{(J_i, \xi_i) : i = 1, \ldots, q\} = \pi \in \Pi(\beta; E)$, we have (using the same notations for elements of R and for the corresponding elements of $C_\infty(\Omega)$):

$$0 \le \left|\sum_{i\in L} \mu(J_i)\, f(\xi_i) - \sum_{i\in L} (H_{\mathcal{B}}) \int_{J_i} f\right|(\omega) \le p_\beta(\omega)$$

for all $L \subset \{1, 2, \ldots, q\}$ (with the convention that the sum along the empty set of any quantity is zero). Fix now $\omega \in \Omega$. If $p_\beta(\omega) = +\infty$, there is nothing to prove. Suppose that $p_\beta(\omega) \in I\!R$. Let L [resp. L'] be the sets of all indices $i \in \{1, \ldots, q\}$ such that

$$\left[\mu(J_i)\, f(\xi_i) - (H_{\mathcal{B}}) \int_{J_i} f\right](\omega) \ge 0 \ \ [\text{resp. } \le 0].$$

We have:

$$\begin{aligned}
0 &\le \sum_{i=1}^{n} \left|\mu(J_i)\, f(\xi_i) - (H_{\mathcal{B}}) \int_{J_i} f\right|(\omega) \\
&= \sum_{i\in L} \left[\mu(J_i)\, f(\xi_i) - (H_{\mathcal{B}}) \int_{J_i} f\right](\omega) \\
&\quad - \sum_{i\in L'} \left[\mu(J_i)\, f(\xi_i) - (H_{\mathcal{B}}) \int_{J_i} f\right](\omega) \le 2\, p_\beta(\omega).
\end{aligned}$$

From this the assertion follows. $\square$

The following uniform version of Lemma 3.9 also can be proved using Propositions 3.7 and 3.8: indeed it is enough to proceed step by step analogously as in the proof of Lemma 3.9, by replacing f with f_λ and taking into account that the involved p_β's and p_{β_i}'s are independent on the parameter $\lambda \in \Lambda$.

Lemma 3.10 *If a family $(f_\lambda)_{\lambda\in\Lambda}$ of $H_{\mathcal{B}}$-integrable functions $f_\lambda : E \to R$ is uniformly $H_{\mathcal{B}}$-integrable on E, so that there exists an (o)-net $(p_\beta)_{\beta\in\mathcal{B}}$ such that (12) holds for all $\lambda \in \Lambda$ and F_λ is the indefinite $H_{\mathcal{B}}$-integral of f_λ, then for the same (o)-net $(p_\beta)_{\beta\in\mathcal{B}}$*

$$\sup\left\{\sum_{(I,x)\in\pi} |\,\mu(I)\, f_\lambda(x) - F_\lambda(I)| : \pi \in \Pi(\beta; E)\right\} \le p_\beta \tag{18}$$

for all $\lambda \in \Lambda$.

4 Variational integral and SL-integral

Having fixed a set $W \subset X$, a complete basis $\mathcal{B}$ and a point-set function $F : \mathcal{I} \times X \to R$ we define, for each $E \subset W$,

$$Var\,(\beta, F, E) = \sup_{\pi} \sum_{(I,x)\in\pi} |F(I, x)|$$

where the involved supremum is taken over the totality of all β-partitions π on E. We also define

$$V\,(\mathcal{B}, F, E) = \inf_{\beta} Var\,(\beta, F, E)\,.$$

Being considered as a set function on the family of all the subsets $E \subset W$, we call $Var\,(\beta, F, \cdot\,)$ the *β-variation* and $V\,(\mathcal{B}, F, \cdot\,)$ the *variational measure* on W, generated by F, with respect to the basis $\mathcal{B}$. (We are using the term "measure" here because in the real-valued case the variational measure is in fact a metric outer measure.) If in the above definitions we replace a complete basis $\mathcal{B}$ by the almost complete basis $\mathcal{B}_0$ generated by $\mathcal{B}$, we get the *essential β-variation* and the *essential variational measure*, respectively. That is

$$Var_{ess}\,(\beta, F, E) = \sup_{\pi} \sum_{(I,x)\in\pi} |F(I, x)|,$$

where the involved supremum is taken over the totality of all β-partitions π on E, $\beta \in \mathcal{B}_0$, and

$$V_{ess}\,(\mathcal{B}, F, E) = \inf_{\beta\in\mathcal{B}_0} Var_{ess}\,(\beta, F, E)\,.$$

It is clear that

$$V_{ess}\,(\mathcal{B}, F, E) \le V\,(\mathcal{B}, F, E)\,. \tag{19}$$

Definition 4.1 We say that two interval-point functions F and G are *variationally equivalent* if $V(\mathcal{B}, F-G, E) = 0$.

Definition 4.2 We say that a function $f : E \to R$ defined on a $\mathcal{B}$-interval E is *variationally integrable* (*$VH_{\mathcal{B}}$-integrable*) if there exists an additive $\mathcal{B}$-interval function τ which is variationally equivalent to the interval-point function $\mu(I)\, f(x)$. In this case we write $(VH_{\mathcal{B}}) \displaystyle\int_E f = \tau(E)$.

As an immediate consequence of the Saks-Henstock lemma we get the following

Theorem 4.3 *For functions $f : E \to R$ defined on a $\mathcal{B}$-interval E the variational integral $VH_{\mathcal{B}}$ is equivalent to the $H_{\mathcal{B}}$-integral.*

Definition 4.4 We say that two interval-point functions F and G are *almost variationally equivalent* if $V_{ess}(\mathcal{B}, F-G, E) = 0$.

It follows from (19) that variational equivalence implies almost variational equivalence.

Now we introduce the concept of SL-integral with respect to an abstract basis $\mathcal{B}$ (in the case of the full interval basis this integral was considered in [9] and [10] for real-valued functions and in [2] for Riesz-space-valued functions). We shall use below two versions of the notion of absolute continuity of the variational measure.

Definition 4.5 We say that the variational measure $V(\mathcal{B}, F, \cdot)$ is *absolutely continuous with respect to a measure μ* or *μ-absolutely continuous on E,* if $V(\mathcal{B}, F, N) = 0$ whenever $\mu(N) = 0$ for $N \subset E$.

Definition 4.6 We say that the variational measure $V(\mathcal{B}, F, \cdot)$ is *uniformly absolutely continuous with respect to a measure μ* or *uniformly μ-absolutely continuous* on E, if there exists an (o)-net $(p_\beta)_\beta$ of elements of R, such that

$$\sup\{Var(\beta, F, N) : N \subset E, \mu(N) = 0\} \leq p_\beta \tag{20}$$

for all $\beta \in \mathcal{B}$.

Definition 4.7 We say that a $\mathcal{B}$-interval R-valued function τ is *of class* (SL) [*of class* (uSL)] or has *property* (SL) [*property* (uSL)] on a $\mathcal{B}$-interval E if the variational measure generated by this function is μ-absolutely continuous [uniformly μ-absolutely continuous] on E.

Definition 4.8 Let R be any Dedekind complete Riesz space. We say that $f : E \to R$ is *SL-integrable* [*uSL-integrable*] on a $\mathcal{B}$-interval E if there exists a $\mathcal{B}$-interval R-valued additive function τ of class (SL)

[of class (uSL)] which is almost variationally equivalent to the interval-point function $\mu(I)\, f(x)$. The function τ is called the *indefinite SL-integral* [*uSL-integral*] of f. In this case we put (*by definition*)

$$(SL)\int_E f = \tau(E)\ \left[(uSL)\int_E f = \tau(E)\right].$$

Of course in the real-valued case uSL-integral coincides with SL-integral.

Proposition 4.9 *If a function f is $H_{\mathcal{B}}$-integrable and its indefinite $H_{\mathcal{B}}$-integral is of class (SL) [of class (uSL)], then f is SL-integrable [uSL-integrable] with F being its indefinite SL-integral [uSL-integral].*

Proof: It follows from the fact that variational equivalence implies almost variational equivalence. $\square$

Proposition 4.10 *Let R be any Dedekind complete Riesz space, $N \subset X$ be a set with $\mu(N) = 0$, and $f_0 : X \to R$ be such that $f_0(x) = 0$ for all $x \notin N$. Then f_0 is SL-integrable and the identically zero function is its indefinite SL-integral.*

Proof: Straightforward. $\square$

As a consequence we get

Proposition 4.11 *Let R be as in Proposition 4.10, and $f, g : X \to R$ be two functions, which differ only on a set of measure μ zero. Then f is (SL)-integrable if and only if g does, and in this case*

$$(SL)\int_X f = (SL)\int_X g.$$

The SL-integral in general does not coincide with the $H_{\mathcal{B}}$-integral. An example of a function which is SL-integrable but is not $H_{\mathcal{B}}$-integrable can be obtained from Example 4.21 given in [2]. In this example the space $R \subset \mathbb{R}^{\mathbb{N}}$ consisting of all sequences with only a finite number of nonzero coordinates (see also [11], p. 479) does not satisfy property σ. An example in the opposite direction also can be constructed using the same property (see below). So this property is a necessary condition for the equivalence of SL- and $H_{\mathcal{B}}$-integrals.

Example 4.12 Consider the dyadic intervals $\Delta_j^k = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right)$, $0 \le j \le 2^k - 1$, $k \in \mathbb{N} \cup \{0\}$. Note that $(0,1) = \cup_{k=1}^{\infty} \Delta_1^k$ and $\Delta_1^k = \Delta_2^{k+1} \cup \Delta_3^{k+1}$. We define a basis $\mathcal{B}$ on $[0,1)$ as follows. Let $\mathcal{I}$ consist of right-open intervals $I = [u,v)$ such that: if $u = 0$, then $I = \Delta_0^k$, $k \in \mathbb{N} \cup \{0\}$; if $u \in \Delta_j^{k+1}$, $j = 2$, or 3, then $I \subset \Delta_j^{k+1}$ with $j = 2$ or 3, respectively. We consider a family of gages δ, each of which is

constant on every interval Δ_j^{k+1}, $j = 2, 3$, $k \in I\!N$. Now every basis set of $\mathcal{B}$ is defined by a gage from the above family as

$$\beta_\delta = \{(I, x) : I = [u, v) \in \mathcal{I},\ x = u,\ |I| \leq \delta(x)\},$$

where the symbol $|\cdot|$ denotes the Lebesgue measure. It is easy to check that this basis has the partitioning property and so the $H_\mathcal{B}$-integral is defined for it. Now let R be a Riesz space which does not satisfy property σ (for example, the space given in [2], Example 4.21), and let a sequence $(p_k)_k$ in R with $p_k \geq 0\ \forall\, k \in I\!N$ be such that for any sequence $(\lambda_k)_k$ of positive real numbers the sequence $(\lambda_k p_k)_k$ is not bounded in R. We define a function $f : [0, 1) \to R$ as follows: $f(x) = 0$ if $x = 0$; $f(x) = (-1)^j p_k$ if $x \in \Delta_j^{k+1}$, $j = 2, 3$, $k \in I\!N$. It is easy to check that with the considered basis the function f is $H_\mathcal{B}$-integrable. Moreover,

$$(H_\mathcal{B}) \int_{\Delta_0^k} f = 0 \text{ for any } k \in I\!N \cup \{0\} \text{ and}$$

$$(H_\mathcal{B}) \int_I f = |I|\, f(x) \text{ for any } I = [u, v) \text{ with } x = u \neq 0.$$

In particular, denoting by F the indefinite integral of f, we have

$$F(I) = (H_\mathcal{B}) \int_I f = |I|\, p_k$$

if $I \subset \Delta_2^{k+1}$. Now take a countable set $E = \{x_k : k \in I\!N\}$ with $x_k \in \Delta_2^{k+1}\ \forall\, k$. Note that for any $\{(I_k, x_k)\}_k = \pi \in \Pi(\beta_\delta; E)$ we have $I_k \subset \Delta_2^{k+1}$ and so

$$\sum_{(I,x)\in\pi} |F(I, x)| = \sum_{(I,x)\in\pi} |I|\, p_k.$$

Because of the choice of the sequence $(p_k)_k$ the set of the values of these sums is not bounded. Hence $Var\,(\beta_\delta, F, E) = V\,(\mathcal{B}, F, E) = +\infty$ and so F is not of class (SL).

We give now some kind of indirect characterization of the space R for which $H_\mathcal{B}$-integrability implies SL-integrability (or uSL-integrability).

Proposition 4.13 *The indefinite $H_\mathcal{B}$-integral of an R-valued function f which is $H_\mathcal{B}$-integrable on a $\mathcal{B}$-interval E is*

i) *of class SL on E if and only if for any set $N \subset E$ with $\mu(N) = 0$ the function $f\chi_N$ is $H_\mathcal{B}$-integrable on E with integral equal to zero;*

ii) *of class uSL on E if and only if the family $\{f\chi_N : N \subset E, \mu(N) = 0\}$ is uniformly $H_\mathcal{B}$-integrable on E with integral equal to zero.*

Proof: We begin with the "if" part of i). Let f be $H_{\mathcal{B}}$-integrable with the indefinite integral F and let N be any set of measure μ zero. By the assumption $f\chi_N$ is $H_{\mathcal{B}}$-integrable with zero integral. Then $f_1 = f - f\chi_N$ is $H_{\mathcal{B}}$-integrable with F being its indefinite integral. For any partition $\pi \subset \beta[N]$ we have

$$\sum_{(I,x)\in\pi} |\,F(I)| = \sum_{(I,x)\in\pi} |\,\mu(I)\,f_1(x) - F(I)|\,,$$

and so, by Lemma 3.9, $V\,(\mathcal{B}, F, N) = 0$. Hence F is of class (SL).

In the opposite direction, let the indefinite $H_{\mathcal{B}}$-integral of f on E be of class (SL). Then, for any partition $\pi \subset \beta[N]$, we have:

$$\sum_{(I,x)\in\pi} |\,\mu(I)\,f(x)\,\chi_N| \leq \sum_{(I,x)\in\pi} |\,\mu(I)\,f(x) - F(I)| + \sum_{(I,x)\in\pi} |\,F(I)|\,.$$

As the Riemann sum for the function $f\chi_N$ with respect to any partition $\pi \in \Pi(\beta; E)$ coincides with the sum with respect to the respective $\beta[N]$-partition, so taking first supremum over all $\pi \subset \beta[N]$ in the above inequality and then taking infimum over all $\beta \in \mathcal{B}$ we get that $f\chi_N$ is $H_{\mathcal{B}}$-integrable with zero integral value. This completes the proof of the part i).

In the proof of the part ii) it is enough to use Lemma 3.10 instead of Lemma 3.9 and to proceed analogously. $\square$

Theorem 4.14 *An R-valued function f, $H_{\mathcal{B}}$-integrable on E, is*

i) *SL-integrable on E with the same integral value if and only if for any set $N \subset E$ with $\mu(N) = 0$ the function $f\chi_N$ is $H_{\mathcal{B}}$-integrable on E with integral equal to zero;*

ii) *uSL-integrable on E with the same integral value if and only if the family $\{f\chi_N : N \subset E, \mu(N) = 0\}$ is uniformly $H_{\mathcal{B}}$-integrable on E with integral equal to zero.*

Proof: It is enough to apply Propositions 4.13 and 4.9. $\square$

As a corollary of the above theorem we get the following

Theorem 4.15 *i) If a Dedekind complete Riesz space R is such that any R-valued function, which is equal to zero almost everywhere, is $H_{\mathcal{B}}$-integrable with integral equal to zero, then any $H_{\mathcal{B}}$-integrable R-valued function is SL-integrable with the same integral value.*

ii) If a Dedekind complete Riesz space R is such that the family of all R-valued functions, which are equal to zero almost everywhere, is uniformly $H_{\mathcal{B}}$-integrable with integral equal to zero, then any $H_{\mathcal{B}}$-integrable R-valued function is uSL-integrable with the same integral value.

As a very strong assumption imposed on the space R which gives a sufficient condition for any R-valued function, which is equal to zero almost everywhere, to be $H_{\mathcal{B}}$-integrable with integral equal to zero, is the assumption that R has both property σ and the Swartz property.

We consider now conditions under which SL-integrability implies $H_{\mathcal{B}}$-integrability.

Theorem 4.16 *If an R-valued function f is SL-integrable on a $\mathcal{B}$-interval E with SL-integral F and if for any set $N \subset E$ with $\mu(N) = 0$ the function $f\chi_N$ is $H_{\mathcal{B}}$-integrable with integral equal to zero, then f is $H_{\mathcal{B}}$-integrable on E and F is its indefinite $H_{\mathcal{B}}$-integral.*

Proof: Let $(r_\theta)_{\theta\in\mathcal{B}_0}$ be an (o)-net given by SL-integrability of f, such that, for any partition $\pi \subset \theta$,

$$\sum_{(I,x)\in\pi} |\,\mu(I)\,f(x) - F(I)| \leq r_\theta.$$

Each fixed $\theta \in \mathcal{B}_0$ of the form $\beta \setminus \beta[N]$ defines a set N of measure μ zero such that $\theta[N]$ is empty. By assumption the function $f\chi_N$ is $H_{\mathcal{B}}$-integrable to zero and this together with Lemma 3.9 implies the existence of an (o)-net $(p_{\theta,\beta})_{\beta\in\mathcal{B}}$ such that, for any partition $\pi \subset \beta[N]$,

$$\sum_{(I,x)\in\pi} |\,\mu(I)\,f(x)\,\chi_N| \leq p_{\theta,\beta}.$$

Another (o)-net $(s_{\theta,\beta})_{\beta\in\mathcal{B}}$ is defined by the fact that F is of class (SL). With this net, for any partition $\pi \subset \beta[N]$, we get

$$\sum_{(I,x)\in\pi} |\,F(I)| \leq s_{\theta,\beta}.$$

Now take a basis set $\gamma \in \mathcal{B}$ defined by setting

$$\gamma[\{x\}] = \begin{cases} \beta[\{x\}] \text{ if } x \in N, \\ \\ \theta[\{x\}] \cap \beta[\{x\}] \text{ if } x \notin N. \end{cases} \tag{21}$$

(This basis set of $\mathcal{B}$ exists by assumption imposed on the basis, see Section 2). It is clear that the set of such γ's is *cofinal* in $\mathcal{B}$, that is for every $\beta \in \mathcal{B}$ there exists γ as in (21) such that $\gamma \subset \beta$.

Now we can sum up the above estimations getting, for any partition $\pi \subset \gamma$,

$$\sum_{(I,x)\in\pi} |\,\mu(I)\,f(x) - F(I)| \leq \sum_{(I,x)\in\pi, x\in N^c} |\,\mu(I)\,f(x) - F(I)| +$$
$$+ \sum_{(I,x)\in\pi, x\in N} |\,\mu(I)\,f(x)\,| + \sum_{(I,x)\in\pi, x\in N} |\,F(I)| \leq r_\theta + p_{\theta,\beta} + s_{\theta,\beta}.$$

From this and cofinality of the set of our involved γ's in $\mathcal{B}$, it follows that

$$\inf\{(r_\theta + p_{\theta,\beta} + s_{\theta,\beta}) : \theta \in \mathcal{B}_0,\ \beta \in \mathcal{B}\} = 0,$$

and consequently f is $H_{\mathcal{B}}$-integrable with F being its $H_{\mathcal{B}}$-integral. □

5 The Fundamental Theorem of Calculus

We now formulate a version of the Fundamental Formula of the Integral Calculus in our abstract setting. Supposing that $\mu(X) < +\infty$ and $\mathcal{B}$ is a complete basis in X, the following result holds:

Theorem 5.1 *Let $f : X \to R$ and τ be an R-valued additive $\mathcal{B}$-interval function of class (SL) on X. Suppose that there exists a set $N \subset X$ with $\mu(N) = 0$, such that f is the (u)-derivative of τ in $X \setminus N$.*

Then f is SL-integrable, and $(SL)\displaystyle\int_X f = \tau(X)$.

Proof: Let $N^c = X \setminus N$. As f is the (u)-derivative of τ in N^c, then we get the existence of an (o)-net $(p_{\beta[N^c]})_{\beta\in\mathcal{B}}$ such that, for all $\beta \in \mathcal{B}$, if $\pi = \{\,(I_i, x_i) : i = 1, \dots, n\}$ is a $\beta[N^c]$-partition, then

$$\sum_{i=1}^{n} |\mu(I_i)f(x_i) - \tau(I_i)\,| \leq \mu(I_i)\, p_{\beta[N^c]}$$

and so

$$\sup_{\pi\subset\beta[N^c]} \sum_{i=1}^{n} |\mu(I_i)f(x_i) - \tau(I_i)\,| \leq \mu(I_i)\, p_{\beta[N^c]}. \tag{22}$$

As $\beta[N^c] \in \mathcal{B}_0$ where $\mathcal{B}_0$ is the almost complete extension of the basis $\mathcal{B}$, then (22) implies that τ is almost variationally equivalent to $\mu(I)f(x)$. This proves the theorem. $\Box$

Now we state some other versions of the Fundamental Formula of Integral Calculus considered for some special Riesz spaces.

Definition 5.2 A Dedekind complete Riesz space is said to be *regular* if it satisfies property σ and if for each sequence $(r_n)_n$ in R, order convergent to zero, there exists a sequence $(l_n)_n$ of positive real numbers, with $\lim_n l_n = +\infty$, such that the sequence $(l_n r_n)_n$ is order convergent to zero. We note that, if $(X, \mathcal{M}, \mu)$ is a measure space with μ positive, σ-additive and σ-finite, then the spaces $L^0(X, \mathcal{M}, \mu)$ and $L^p(X, \mathcal{M}, \mu)$, with $1 \leq p < +\infty$, are regular; moreover the space of all real sequences, with the usual coordinatewise ordering, is regular (see [11], pp. 478-481, and [3]).

The following two results hold, which can be proved repeating the arguments of Theorems 4.1 and 4.2 of [3] (for similar theorems existing in the literature in other abstract structures see [22], for the real case of the Fundamental Formula of Integral Calculus see also [16], and for other versions in the context of Riesz spaces see [2, 3, 24]).

Theorem 5.3 *Let R be a regular Riesz space, $f : X \to R$ and τ be an R-valued $\mathcal{B}$-interval function, such that there exists a countable set $Q \subset X$, $Q \in \mathcal{M}$, such that f is the (u)-derivative of τ in $X \setminus Q$ and τ is continuous in Q. Then f is $H_{\mathcal{B}}$-integrable on X, and $(H_{\mathcal{B}})\displaystyle\int_X f = \tau(X)$. Moreover, if R is*

any arbitrary Dedekind complete Riesz space, and τ is a $\mathcal{B}$-interval R-valued function, (u)-differentiable in X with derivative τ', then τ' is $H_{\mathcal{B}}$-integrable on X, and $(H_{\mathcal{B}}) \int_X \tau' = \tau(X)$.

Remark 5.4 We observe that all our results, except the ones concerning Theorem 3.1, remain true also if we consider a (finitely additive) Riesz-space-valued positive measure μ instead of a real-valued measure μ: to this aim, it is enough to consider a *product triple* (R_1, R_2, R), whose definition will be given below. Our involved functions f, g and our measure μ will be R_1- and R_2-valued respectively, while our function τ and our integral will be R-valued.

Let R_1, R_2, R be three Dedekind complete Riesz spaces. We say that (R_1, R_2, R) is a *product triple* if there exists a map $\cdot : R_1 \times R_2 \to R$, which we will call *product*, such that

5.4.1) $(r_1 + s_1) \cdot r_2 = r_1 \cdot r_2 + s_1 \cdot r_2$,

5.4.2) $r_1 \cdot (r_2 + s_2) = r_1 \cdot r_2 + r_1 \cdot s_2$,

5.4.3) $[r_1 \geq s_1,\, r_2 \geq 0] \Rightarrow [r_1 \cdot r_2 \geq s_1 \cdot r_2]$,

5.4.4) $[r_1 \geq 0,\, r_2 \geq s_2] \Rightarrow [r_1 \cdot r_2 \geq r_1 \cdot s_2]$ for all $r_j, s_j \in R_j, j = 1, 2$;

5.4.5) if $(a_\lambda)_{\lambda \in \Lambda}$ any (o)-net in R_2 and $0 \leq b \in R_1$, then $(b \cdot a_\lambda)_{\lambda \in \Lambda}$ is an (o)-net in R;

5.4.6) if $(a_\lambda)_{\lambda \in \Lambda}$ is any (o)-net in R_1 and $0 \leq b \in R_2$, then $(a_\lambda \cdot b)_{\lambda \in \Lambda}$ is an (o)-net in R.

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A Harnack Inequality for Solutions of Doubly Nonlinear Parabolic Equations

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Abstract - We consider positive solutions of the doubly nonlinear parabolic equation

$$(|u|^{p-1})_t = \operatorname{div}(|Du|^{p-2}Du), \qquad p > 2.$$

We prove mean value inequalities for positive powers of nonnegative subsolutions and for negative powers of positive supersolutions using De Giorgi's methods. We combine them with Moser's logarithmic estimates to show that positive solutions satisfy a proper Harnack inequality.

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Key words: Nonlinear parabolic equation, mean value inequalities, Harnack estimate.

1. - Introduction

In the final Section of [16], without any explicit calculation, Trudinger states that Moser's method (see [11]) can be extended to prove the following

Theorem 1. *Let u be a positive solution of*

$$(|u|^{p-1})_t - \operatorname{div}(|Du|^{p-2}Du) = 0 \tag{1}$$

in Ω_T and suppose that the cylinder $[(x_0, t_0) + Q(2\rho, 2^p\theta\rho^p)] \subset \Omega_T$. Then there exists a constant $C > 1$ that depends only on the data and on θ s.t.

$$\sup_{[(0,-\theta\rho^p)+Q(\frac{\rho}{2},\frac{\theta}{2^p}\rho^p)]} u \le C \inf_{Q(\frac{\rho}{2},\frac{\theta}{2^p}\rho^p)} u$$

(we refer to the next Section for the notation). It is immediate to see that with the substitution $u^{p-1} = v$, (1) can be rewritten as

$$v_t - \left(\frac{1}{p-1}\right)^{p-1} \operatorname{div}(|v|^{2-p}|Dv|^{p-2}Dv) = 0$$

which is just a particular istance of the more general class of doubly nonlinear parabolic equations

$$v_t - \operatorname{div}(|v|^{m-1}|Dv|^{p-2}Dv) = 0 \tag{2}$$

where $p > 1$ and $m + p > 2$. This equation describes a lot of phenomena. Just to limit ourselves to the motion of fluids in media, when $p = 2$ we obtain the well - known porous medium equation; if $m = 1$ we have the parabolic p-laplacian, which describes the nonstationary flow in a porous medium of fluids with a power

dependance of the tangential stress on the velocity of the displacement under elastic conditions; in the whole generality, that is when $p \neq 2$ and $m \neq 1$, (2) is a model for the polytropic case when we have dependance between stress and velocity of the displacement. However these are just few examples; the interested reader can find further applications in [1] or in [15].

Regularity issues for doubly nonlinear parabolic equations like (2) have been considered by a lot of authors and a complete bibliographic list cannot be given here: under this point of view, we refer to [4] (updated to 1993) and to [5] (just published). Let us just mention that, among others, continuity has been proved both in the degenerate ($p > 2$ and $m > 1$) and in the singular ($1 < p < 2$ or $0 < m < 1$) case in [8], [13] and [18]. Other interesting references can be found in [6], where the regularity in Sobolev spaces is considered.

Coming back to (1), the reason of Trudinger's statement basically lies in the p-homogeneity of the equation, that makes the proof of mean value inequalities for positive and negative powers of the solution as natural as in the case of the general parabolic equation with bounded and measurable coefficient a_{ij}

$$u_t - \operatorname{div}\left(\sum_{j=1}^{N} a_{ij} D_j u\right) = 0.$$

The Harnack inequality has indeed been proved with full details not only for (1), but more generally for (2) when $p > 1$, $m + p > 2$ and $m + p + \frac{p}{N} > 3$ in [17] and the essential tools are the comparison principle, proper L^∞-estimates and the Hölder continuity of u. Now a natural question arises, namely if the particular link between m and p in (1) allows a different method, which does not require any previous knowledge of the regularity of u (not to mention the comparison principle).

The p-homogeneity of (1) naturally suggests an approach based on parabolic De Giorgi classes of order p (see [10], Chapter II, Section 7 for a definition when $p = 2$; in [7] we consider the general case), but it is rather easy to see that they do not correspond to solutions of (1). Hence we have a twofold problem: understand what kind of De Giorgi classes (that is, energy estimates) are associated to positive solutions of (1) and verify if Trudinger's claim can be proved using De Giorgi's method, starting from the corresponding classes. In this short note we characterize the classes associated to (1) and prove that proper mean value inequalities can indeed be obtained relying on them (see Sections 2 and 3). We then conclude in Section 4 with a proof of Theorem 1 based on logarithmic estimates first proved by Moser in [12].

As explained in Remark 1 (see the next Section for more details), our result applies to more general equations and under this point of view it can indeed be seen as a (limited!) extension of the Harnack inequality proved in [17].

When finishing this note, we learnt that T. Kuusi (see [9]) gave a full proof of Trudinger's statement using classical Moser's estimate.

2. - Notation and Energy Inequalities

Let Ω be an open bounded domain in $\mathbf{R}^N$; for $T > 0$ we denote by Ω_T the cylindrical domain $\Omega_T \equiv \Omega \times]0, T]$. In the following we will work with smooth solutions of the equation

$$(|u|^{p-1})_t - \operatorname{div}(|Du|^{p-2} Du) = 0 \qquad \text{in } \Omega_T \tag{3}$$

with $p > 2$, but our estimates depend only on the data and not on the smoothness of the solutions, which is assumed just in order to simplify some calculations. Moreover we will deal with bounded nonnegative solutions, namely we assume that

$$\|u\|_{L^\infty(\Omega_T)} \leq M, \qquad u(x,t) \geq 0 \quad \forall\, (x,t) \in \Omega_T$$

so that we can drop the modulus in the $|u|^{p-1}$ term. For $\rho > 0$ denote by K_ρ the ball of radius ρ centered at the origin, i.e.

$$K_\rho \equiv \{x \in \mathbf{R}^N \mid |x| < \rho\}.$$

We let $[y + K_\rho]$ denote the ball centered at y and congruent to K_ρ, i.e.

$$[y + K_\rho] \equiv \{x \in \mathbf{R}^N | \ |x - y| < \rho\}.$$

For $\theta > 0$ denote by $Q(\rho, \theta\rho^p)$ the cylinder of cross section K_ρ, height $\theta\rho^p$ and vertex at the origin, i.e.

$$Q(\rho, \theta\rho^p) \equiv K_\rho \times] - \theta\rho^p, 0].$$

For a point $(y, s) \in \mathbf{R}^{N+1}$ we let $[(y, s) + Q(\rho, \theta\rho^p)]$ be the cylinder of vertex at (y, s) and congruent to $Q(\rho, \theta\rho^p)$, i.e.

$$[(y, s) + Q(\rho, \theta\rho^p)] \equiv [y + K_\rho] \times]s - \theta\rho^p, s].$$

The truncations $(u - k)_+$ and $(u - k)_-$ for $k \in \mathbf{R}$ are defined by

$$(u - k)_+ \equiv \max\{u - k, 0\}; \qquad (u - k)_- \equiv \{k - u, 0\}$$

and we set

$$A^{\pm}_{k,\rho}(\tau) \equiv \{x \in K_\rho : \ (u - k)_\pm(x, \tau) > 0\}.$$

In the following with $|\Sigma|$ we denote the Lebesgue measure of a measurable set Σ.

Remark 1. *Even if we consider the p-Laplacian operator, all the following results still hold if we deal with a second order homogeneous operator*

$$\mathcal{L} = \operatorname{div} \mathbf{a}(x, t, u, Du)$$

where $\mathbf{a} : \Omega_T \times \mathbf{R}^{N+1} \to \mathbf{R}^N$ *is measurable and for a.e.* $(x, t) \in \Omega_T$ *satisfies*

$$\mathbf{a}(x, t, u, Du) \cdot Du \geq C_1 |Du|^p,$$

$$|\mathbf{a}(x, t, u, Du)| \leq C_2 |Du|^{p-1}$$

for two given constant $0 < C_1 < C_2$. *The main point is that the lower order terms are zero* ■.

Definition 1. *A measurable function u is a* <u>*local*</u> *weak sub (super) - solution of (3) if*

$$u \in C^0_{loc}(0, T; L^p_{loc}(\Omega)) \cap L^p_{loc}(0, T; W^{1,p}_{loc}(\Omega))$$

and for every compact subset $\mathcal{K}$ *of* Ω *and for every subinterval* $[t_1, t_2]$ *of* $]0, T]$ *we have that*

$$\int_{\mathcal{K}} u^{p-1}\zeta \, dx|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathcal{K}} [-u^{p-1}\zeta_t + \sum_{i=1}^{N} |Du|^{p-2} D_i u D_i \zeta] \, dx d\tau \leq (\geq) 0 \tag{4}$$

for all testing function

$$\zeta \in W^{1,p}_{loc}(0, T; L^p(\mathcal{K})) \cap L^p_{loc}(0, T; W^{1,p}_0(\mathcal{K}))$$

with $\zeta \geq 0$. *A function u that is both a local subsolution and a local supersolution is a local solution* ■.

For general degenerate or singular parabolic equations of the type considered in [4], energy inequalities are proved both for $(u - k)_+$ and $(u - k)_-$ with $k \in \mathbf{R}$. Due to the presence of the $(u^{p-1})_t$ term, which gives rise to some difficulties when dealing with $(u - k)_-$, here we follow a different strategy in that we prove energy inequalities for +-truncations of u and $\frac{1}{u}$. The fact that we deal with $(\frac{1}{u} - k)_+$ instead of working with $(u - k)_-$ should not look so surprising as both are convex, monotone decreasing function of the argument u. For the sake of simplicity we state and prove the two energy inequalities indipendently from one another.

Proposition 1 (First Local Energy Estimate). *Let u be a locally bounded nonnegative weak subsolution of (3) in Ω_T. There exists a constant γ that can be determined a priori in terms of the data such that for every cylinder $[(y,s)+Q(\rho,\theta\rho^p)]\subset\Omega_T$ and for every $k\in\mathbf{R}_+$*

$$
\begin{aligned}
&\frac{p-1}{p}\sup_{s-\theta\rho^p<t<s}\int_{[y+K_\rho]}(u-k)_+^p\varphi^p(x,t)\,dx+\iint_{[(y,s)+Q(\rho,\theta\rho^p)]}|D(u-k)_+|^p\varphi^p\,dxd\tau\\
&\leq\gamma\Bigg[\iint_{[(y,s)+Q(\rho,\theta\rho^p)]}(u-k)_+^p|D\varphi|^p\,dxd\tau+\\
&+\iint_{[(y,s)+Q(\rho,\theta\rho^p)]\cap\{u-k<k\}}p(p-1)\sum_{n=0}^{\infty}\binom{p-2}{n}k^{p-2-n}\frac{(u-k)_+^{n+2}}{n+2}\varphi^{p-1}\varphi_t\,dxd\tau+\\
&+\iint_{[(y,s)+Q(\rho,\theta\rho^p)]\cap\{u-k>k\}}p(p-1)\sum_{n=0}^{\infty}\binom{p-2}{n}k^{n}\frac{(u-k)_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau+\\
&+\iint_{[(y,s)+Q(\rho,\theta\rho^p)]\cap\{u-k=k\}}(p-1)2^{p-2}(u-k)_+^p\varphi^{p-1}\varphi_t\,dxd\tau\Bigg]
\end{aligned}
\tag{5}
$$

for every $\varphi\in\mathcal{C}([(y,s)+Q(\rho,\theta\rho^p)])$ with $\varphi(\cdot,s-\theta\rho^p)=0$ where $\mathcal{C}(Q(\rho,\theta\rho^p))$ denotes the class of all piecewise smooth functions $\varphi:Q(\rho,\theta\rho^p)\to\mathbf{R}^+$ such that
1) $x\to\varphi(x,t)\in W_0^{1,\infty}(K_\rho)\quad\forall t\in]-\theta\rho^p,0]$;
2) $\varphi_t\geq 0$;
3) $|D\varphi|+\varphi_t\in L^\infty(Q(\rho,\theta\rho^p))$.

Proof - Since we assume u regular, we can rewrite (4) in a slightly different way, namely

$$\int_{t_1}^{t_2}\int_{\mathcal{K}}[(u^{p-1})_t\zeta+\sum_{i=1}^N|Du|^{p-2}D_iuD_i\zeta]\,dxd\tau\leq 0. \tag{6}$$

After a translation we can assume $(y,s)\equiv(0,0)$ without loss of generality. Let us now fix $k\in\mathbf{R}_+$ and take $\zeta=(u-k)_+\varphi^p$ with $\varphi\in\mathcal{C}(Q(\rho,\theta\rho^p))$ and $\varphi(\cdot,-\theta\rho^p)=0$ as test function in (6) and integrate over $K_\rho\times]-\theta\rho^p,t]$ with $t\in]-\theta\rho^p,0]$. We obtain

$$\iint_{Q(\rho,\theta\rho^p+t)}(u^{p-1})_t(u-k)_+\varphi^p\,dxd\tau+\iint_{Q(\rho,\theta\rho^p+t)}|Du|^{p-2}Du\cdot D((u-k)_+\varphi^p)\,dxd\tau=0$$

where $Q(\rho,\theta\rho^p+t)=K_\rho\times]-\theta\rho^p,t]\subseteq Q(\rho,\theta\rho^p)$. As usual

$$
\begin{aligned}
\iint_{Q(\rho,\theta\rho^p+t)}|Du|^{p-2}Du\cdot D((u-k)_+\varphi^p)\,dxd\tau&=\iint_{Q(\rho,\theta\rho^p+t)}\varphi^p|D(u-k)_+|^p\,dxd\tau+\\
&+p\iint_{Q(\rho,\theta\rho^p+t)}|D(u-k)_+|^{p-2}\varphi^{p-1}(u-k)_+D(u-k)_+\cdot D\varphi\,dxd\tau
\end{aligned}
$$

and for the estimate of the second term of the right - hand side we reason as usual. Let us now come to the estimate of $\iint_{Q(\rho,\theta\rho^p+t)}(u^{p-1})_t(u-k)_+\varphi^p\,dxd\tau$. Relying on the series expansion of $(1+z)^\alpha$ we have

$$
(u^{p-1})_t=\begin{cases}
(p-1)k^{p-2}\displaystyle\sum_{n=0}^{\infty}\binom{p-2}{n}\left(\frac{u-k}{k}\right)^n u_t & \text{if } 0<u-k<k\\[2ex]
(p-1)(u-k)^{p-2}\displaystyle\sum_{n=0}^{\infty}\binom{p-2}{n}\left(\frac{k}{u-k}\right)^n u_t & \text{if } u-k>k>0\\[2ex]
(p-1)2^{p-2}(u-k)^{p-2}u_t & \text{if } u-k=k>0
\end{cases}
\tag{7}
$$

and hence

$$\iint_{Q(\rho,\theta\rho^p+t)} (u^{p-1})_t (u-k)_+ \varphi^p \, dx d\tau =$$

$$= (p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}} k^{p-2} \sum_{n=0}^{\infty} \binom{p-2}{n} \left(\frac{u-k}{k}\right)^n u_t (u-k)_+ \varphi^p \, dx d\tau + \tag{8}$$

$$+ (p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}} (u-k)_+^{p-2} \sum_{n=0}^{\infty} \binom{p-2}{n} \left(\frac{k}{(u-k)_+}\right)^n u_t (u-k)_+ \varphi^p \, dx d\tau + \tag{9}$$

$$+ (p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k=k\}} 2^{p-2} (u-k)_+^{p-1} u_t \varphi^p \, dx d\tau. \tag{10}$$

We clearly need to work distinctly on the previous three terms. Let us start from (8). We have

$$(p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}} k^{p-2} \sum_{n=0}^{\infty} \binom{p-2}{n} \left(\frac{u-k}{k}\right)^n u_t (u-k)_+ \varphi^p \, dx d\tau =$$

$$= (p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^{p-2-n} (u-k)_+^{n+1} u_t \, \varphi^p \, dx d\tau =$$

$$= (p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^{p-2-n} \left[\frac{(u-k)_+^{n+2}}{n+2}\right]_t \varphi^p \, dx d\tau =$$

$$= (p-1) \int_{K_\rho\cap\{u-k<k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^{p-2-n} \frac{(u-k)_+^{n+2}}{n+2} \varphi^p(x,t) \, dx +$$

$$- p(p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^{p-2-n} \frac{(u-k)_+^{n+2}}{n+2} \varphi^{p-1} \varphi_t \, dx d\tau.$$

Let us now deal with (9). We obtain

$$(p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}} (u-k)^{p-2} \sum_{n=0}^{\infty} \binom{p-2}{n} \left(\frac{k}{(u-k)_+}\right)^n u_t (u-k)_+ \varphi^p \, dx d\tau =$$

$$= (p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^n (u-k)_+^{p-1-n} u_t \varphi^p \, dx d\tau =$$

$$= (p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^n \left[\frac{(u-k)_+^{p-n}}{p-n}\right]_t \varphi^p \, dx d\tau =$$

$$= (p-1) \int_{K_\rho\cap\{u-k>k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^n \frac{(u-k)_+^{p-n}}{p-n} \varphi^p(x,t) \, dx +$$

$$- p(p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^n \frac{(u-k)_+^{p-n}}{p-n} \varphi^{p-1} \varphi_t \, dx d\tau.$$

Finally, coming to (10) we get

$$(p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k=k\}} 2^{p-2} (u-k)_+^{p-1} u_t \varphi^p \, dx d\tau = (p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k=k\}} 2^{p-2} \left[\frac{(u-k)_+^p}{p}\right]_t \varphi^p \, dx d\tau =$$

$$= \frac{p-1}{p} \int_{K_\rho\cap\{u-k=k\}} 2^{p-2} (u-k)_+^p \varphi^p(x,t) \, dx - (p-1) \iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k=k\}} 2^{p-2} (u-k)_+^p \varphi^{p-1} \varphi_t \, dx d\tau.$$

If we now put everything together we obtain

$$
\begin{aligned}
&(p-1)\int_{K_\rho\cap\{u-k<k\}}\sum_{n=0}^{\infty}\binom{p-2}{n}k^{p-2-n}\frac{(u-k)_+^{n+2}}{n+2}\,\varphi^p(x,t)\,dx+\\
&+(p-1)\int_{K_\rho\cap\{u-k>k\}}\sum_{n=0}^{\infty}\binom{p-2}{n}k^{n}\frac{(u-k)_+^{p-n}}{p-n}\varphi^p(x,t)\,dx+\\
&+\frac{p-1}{p}\int_{K_\rho\cap\{u-k=k\}}2^{p-2}(u-k)_+^p\varphi^p(x,t)\,dx+\iint_{Q(\rho,\theta\rho^p+t)}\varphi^p|D(u-k)_+|^p\,dxd\tau\le\\
&\le\gamma\Bigg[\iint_{Q(\rho,\theta\rho^p+t)}(u-k)_+^p|D\varphi|^p\,dxd\tau+\\
&+p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}}\sum_{n=0}^{\infty}\binom{p-2}{n}k^{p-2-n}\frac{(u-k)_+^{n+2}}{n+2}\,\varphi^{p-1}\varphi_t\,dxd\tau+\\
&+p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}}\sum_{n=0}^{\infty}\binom{p-2}{n}k^{n}\frac{(u-k)_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau+\\
&+(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k=k\}}2^{p-2}(u-k)_+^p\varphi^{p-1}\varphi_t\,dxd\tau\Bigg].
\end{aligned}
$$

If $0<u-k<k$ then

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(p-1)\binom{p-2}{n}k^{p-2-n}\frac{(u-k)_+^{n+2}}{n+2}=\\
&=(p-1)\left[\binom{p-2}{0}k^{p-2}\frac{(u-k)_+^2}{2}+\binom{p-2}{1}k^{p-3}\frac{(u-k)_+^3}{3}+\dots\right]\ge\frac{p-1}{2}(u-k)_+^p
\end{aligned}
$$

as $\binom{p-2}{1}>0$. Analogously, if $u-k>k>0$

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(p-1)\binom{p-2}{n}k^{n}\frac{(u-k)_+^{p-n}}{p-n}=\\
&=(p-1)\left[\binom{p-2}{0}\frac{(u-k)_+^p}{p}+\binom{p-2}{1}\frac{(u-k)_+^{p-1}}{p-1}k+\dots\right]\ge\frac{p-1}{p}(u-k)_+^p.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&\frac{p-1}{p}\int_{K_\rho}(u-k)_+^p\varphi^p(x,t)\,dx+\iint_{Q(\rho,\theta\rho^p+t)}\varphi^p|D(u-k)_+|^p\,dxd\tau\le\\
&\le\gamma\Bigg[\iint_{Q(\rho,\theta\rho^p+t)}(u-k)_+^p|D\varphi|^p\,dxd\tau+\\
&+p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}}\sum_{n=0}^{\infty}\binom{p-2}{n}k^{p-2-n}\frac{(u-k)_+^{n+2}}{n+2}\,\varphi^{p-1}\varphi_t\,dxd\tau+\\
&+p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}}\sum_{n=0}^{\infty}\binom{p-2}{n}k^{n}\frac{(u-k)_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau+\\
&+(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k=k\}}2^{p-2}(u-k)_+^p\varphi^{p-1}\varphi_t\,dxd\tau\Bigg]
\end{aligned}
$$

and since $t \in]-\theta\rho^p, 0]$ is arbitrary, we conclude ■.

Proposition 2 (Second Local Energy Estimate). *Let u be a locally bounded positive weak supersolution of (3) in Ω_T and let us set $v = \frac{1}{u}$. There exists a constant γ that can be determined a priori in terms of the data such that for every cylinder $[(y,s) + Q(\rho, \theta\rho^p)] \subset \Omega_T$ and for every $l \in \mathbf{R}_+$*

$$\frac{p-1}{p} \sup_{s-\theta\rho^p < t < s} \int_{[y+K_\rho]} (v-l)_+^p \varphi^p(x,t)\, dx + \iint_{[(y,s)+Q(\rho,\theta\rho^p)]} |D(v-l)_+|^p \varphi^p \, dx d\tau$$

$$\leq \gamma \left[\iint_{[(y,s)+Q(\rho,\theta\rho^p)]} (v-l)_+^p |D\varphi|^p \, dx d\tau + \right.$$

$$+ \iint_{[(y,s)+Q(\rho,\theta\rho^p)] \cap \{v-l<l\}} p(p-1) \sum_{n=0}^{\infty} \binom{p-2}{n} l^{p-2-n} \frac{(v-l)_+^{n+2}}{n+2} \varphi^{p-1} \varphi_t \, dx d\tau + \tag{11}$$

$$+ \iint_{[(y,s)+Q(\rho,\theta\rho^p)] \cap \{v-l>l\}} p(p-1) \sum_{n=0}^{\infty} \binom{p-2}{n} l^{n} \frac{(v-l)_+^{p-n}}{p-n} \varphi^{p-1} \varphi_t \, dx d\tau +$$

$$\left. + \iint_{[(y,s)+Q(\rho,\theta\rho^p)] \cap \{v-l=l\}} (p-1) 2^{p-2} (v-l)_+^p \varphi^{p-1} \varphi_t \, dx d\tau \right]$$

for every $\varphi \in \mathcal{C}([(y,s) + Q(\rho, \theta\rho^p)])$ with $\varphi(\cdot, s - \theta\rho^p) = 0$.

Proof - After a translation we can assume $(y,s) \equiv (0,0)$ without loss of generality. If we set $v = \frac{1}{u}$, (6) becomes

$$\int_{t_1}^{t_2} \int_{\mathcal{K}} [(p-1) \frac{1}{v^p} v_t \zeta + \sum_{i=1}^{N} \frac{|Dv|^{p-2}}{v^{2p-2}} D_i v D_i \zeta] \, dx d\tau \leq 0. \tag{12}$$

Let us now fix $l \in \mathbf{R}_+$ and take $\zeta = (v-l)_+ v^{2p-2} \varphi^p$ with $\varphi \in \mathcal{C}(Q(\rho, \theta\rho^p))$ and $\varphi(\cdot, -\theta\rho^p) = 0$ as test function in (12) and integrate over $K_\rho \times]-\theta\rho^p, t]$ with $t \in]-\theta\rho^p, 0]$. With simple calculations we obtain

$$\iint_{Q(\rho,\theta\rho^p+t)} (p-1)(v-l)_+ v^{p-2} v_t \varphi^p \, dx d\tau + \iint_{Q(\rho,\theta\rho^p+t)} |Dv|^{p-2} (Dv \cdot D(v-l)_+) \varphi^p \, dx d\tau +$$

$$+ \iint_{Q(\rho,\theta\rho^p+t)} (2p-2) \frac{|Dv|^p}{v} (v-l)_+ \varphi^p \, dx d\tau + \iint_{Q(\rho,\theta\rho^p+t)} p |Dv|^{p-2} (v-l)_+ \varphi^{p-1} Dv \cdot D\varphi \, dx d\tau \leq 0$$

that is

$$\iint_{Q(\rho,\theta\rho^p+t)} (p-1)(v-l)_+ v^{p-2} v_t \varphi^p \, dx d\tau + \iint_{Q(\rho,\theta\rho^p+t)} |D(v-l)_+|^p \varphi^p \, dx d\tau +$$

$$+ \iint_{Q(\rho,\theta\rho^p+t)} (2p-2) |D(v-l)_+|^p \frac{(v-l)_+}{v} \varphi^p \, dx d\tau +$$

$$+ \iint_{Q(\rho,\theta\rho^p+t)} p |D(v-l)_+|^{p-2} (v-l)_+ \varphi^{p-1} D(v-l)_+ \cdot D\varphi \, dx d\tau \leq 0.$$

We can then work as in the proof of the previous Proposition to conclude that

$$\frac{p-1}{p} \int_{K_\rho} (v-l)_+^p \varphi^p(x,t) \, dx + \iint_{Q(\rho,\theta\rho^p+t)} \varphi^p |D(v-l)_+|^p \, dx d\tau +$$

$$+ (2p-2) \iint_{Q(\rho,\theta\rho^p+t)} \frac{(v-l)_+}{v} |D(v-l)_+|^p \varphi^p \, dx d\tau \leq \gamma \left[\iint_{Q(\rho,\theta\rho^p+t)} (v-l)_+^p |D\varphi|^p \, dx d\tau + \right.$$

$$+p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{v-l<l\}}\sum_{n=0}^{\infty}\binom{p-2}{n}l^{p-2-n}\frac{(v-l)_+^{n+2}}{n+2}\varphi^{p-1}\varphi_t\,dxd\tau+$$

$$+p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{v-l>l\}}\sum_{n=0}^{\infty}\binom{p-2}{n}l^{n}\frac{(v-l)_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau+$$

$$\left.+(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{v-l=l\}}2^{p-2}(v-l)_+^{p}\varphi^{p-1}\varphi_t\,dxd\tau\right].$$

The third term on the left - hand side can be dropped since it is positive and relying on the arbitrarness of t we are finished ∎.

3. - Mean Value Inequalities for Sub- and Supersolutions

As we discussed in the first Section, in [16] Trudinger states that it can be proved that positive solutions of (3) satisfy proper mean value inequalities relying on the method first developed in [11]. Here we show that the same results can be proved using De Giorgi's technique based on the energy inequalities for truncated subsolutions of the previous Section. We have

Proposition 3. *Let u be a nonnegative subsolution of (3) Then for all $\epsilon\in]0,p]$ there exists a positive constant C depending upon the data, θ and ϵ s. t. for all $[(x_0,t_0)+Q(\rho,\theta\rho^p)]\subset\Omega_T$ and for all $\sigma\in]0,1[$*

$$\sup_{[(x_0,t_0)+Q(\sigma\rho,\theta\sigma^p\rho^p)]}u\le\frac{C}{(1-\sigma)^{\frac{N+p}{\epsilon}}}\left(\fint\!\!\fint_{[(x_0,t_0)+Q(\rho,\theta\rho^p)]}|u|^{\epsilon}\,dxd\tau\right)^{\frac{1}{\epsilon}}\quad\blacksquare. \tag{13}$$

Proposition 4. *Let u be a positive supersolution of (3). Then for all $\epsilon\in]0,p]$ there exists a positive constant D depending upon the data, θ and ϵ s. t. for all $[(x_0,t_0)+Q(\rho,\theta\rho^p)]\subset\Omega_T$ and for all $\sigma\in]0,1[$ the function $v=\frac{1}{u}$ satisfies*

$$\sup_{[(x_0,t_0)+Q(\sigma\rho,\theta\sigma^p\rho^p)]}v\le\frac{D}{(1-\sigma)^{\frac{(N+p)}{\epsilon}}}\left(\fint\!\!\fint_{[(x_0,t_0)+Q(\rho,\theta\rho^p)]}|v|^{\epsilon}\,dxd\tau\right)^{\frac{1}{\epsilon}}\quad\blacksquare. \tag{14}$$

Proof - Due to the same structure of (5) and (11), we limit ourselves to the proof of (13). We assume $k>0$ and set

$$k_j=k(1-\frac{1}{2^j}).$$

As usual we can suppose that $(x_0,t_0)=(0,0)$. Let us now consider the second term on the right - hand side of (5) with respect to level k_{j+1}.

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}p(p-1)\sum_{n=0}^{\infty}\binom{p-2}{n}k_{j+1}^{p-2-n}\frac{(u-k_{j+1})_+^{n+2}}{n+2}\varphi^{p-1}\varphi_t\,dxd\tau=$$

$$=\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}p(p-1)\sum_{n=0}^{[p-2]}\binom{p-2}{n}k_{j+1}^{p-2-n}\frac{(u-k_{j+1})_+^{n+2}}{n+2}\varphi^{p-1}\varphi_t\,dxd\tau+$$

$$+\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}p(p-1)\sum_{n=[p-2]+1}^{\infty}\binom{p-2}{n}k_{j+1}^{p-2-n}\frac{(u-k_{j+1})_+^{n+2}}{n+2}\varphi^{p-1}\varphi_t\,dxd\tau.$$

Notice that $\forall\, s>0$

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}(u-k_j)_+^s\,dxd\tau=\iint_{Q(\rho,\theta\rho^p)\cap\{k_j<u<2k_{j+1}\}}(u-k_j)_+^s\,dxd\tau\ge$$

$$\geq \iint_{Q(\rho,\theta\rho^p)\cap\{k_{j+1}<u<2k_{j+1}\}} (u-k_j)_+^s \, dxd\tau \geq (k_{j+1}-k_j)^s |A_{j+1}| = \frac{k^s}{2^{(j+1)s}} |A_{j+1}| \tag{15}$$

where $A_{j+1} = \{k_{j+1} < u < 2k_{j+1}\}$. Then

$$p(p-1) \sum_{n=0}^{[p-2]} \binom{p-2}{n} \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} k_{j+1}^{p-2-n} \frac{(u-k_{j+1})_+^{n+2}}{n+2} \varphi^{p-1}\varphi_t \, dxd\tau$$

$$\leq p(p-1) \sum_{n=0}^{[p-2]} \binom{p-2}{n} \frac{k_{j+1}^{p-2-n}}{n+2} \left(\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} (u-k_{j+1})_+^p (\varphi^{p-1}\varphi_t)^{\frac{p}{n+2}} \, dxd\tau \right)^{\frac{n+2}{p}} |A_{j+1}|^{1-\frac{n+2}{p}}$$

$$\leq C_\varphi \, p(p-1) \sum_{n=0}^{[p-2]} \binom{p-2}{n} \frac{k_{j+1}^{p-2-n}}{n+2} \left(\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} (u-k_j)_+^p \, dxd\tau \right)^{\frac{n+2}{p}} \cdot$$

$$\cdot \left(\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} (u-k_j)_+^p \, dxd\tau \right)^{1-\frac{n+2}{p}} \frac{2^{(j+1)(p-(n+2))}}{k^{p-(n+2)}}$$

where we have taken (15) into account and $C_\varphi := \sup \varphi^{p-1}\varphi_t$,

$$\leq C_\varphi \, p(p-1) \sum_{n=0}^{[p-2]} \binom{p-2}{n} \frac{2^{(j+1)(p-(n+2))}}{n+2} \left(1 - \frac{1}{2^{j+1}}\right)^{p-(n+2)} \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} (u-k_j)_+^p \, dxd\tau$$

$$\leq C_\varphi \, p(p-1) \sum_{n=0}^{[p-2]} \binom{p-2}{n} \frac{2^{(j+1)(p-(n+2))}}{n+2} \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} (u-k_j)_+^p \, dxd\tau$$

$$\leq \gamma(p) \, 2^{jp} \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} (u-k_j)_+^p \, dxd\tau.$$

Then we have

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} p(p-1) \sum_{n=0}^{\infty} \binom{p-2}{n} k_{j+1}^{p-2-n} \frac{(u-k_{j+1})_+^{n+2}}{n+2} \varphi^{p-1}\varphi_t \, dxd\tau \leq$$

$$\leq \gamma(p) \, 2^{jp} \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} (u-k_j)_+^p \, dxd\tau +$$

$$+ C_\varphi p(p-1) \binom{p-2}{[p-2]+1} k_{j+1}^{p-3-[p-2]} \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} \frac{(u-k_{j+1})_+^{[p-2]+3}}{[p-2]+3} \, dxd\tau.$$

Moreover

$$0 < (u-k_{j+1})_+ < k_{j+1} \quad \Rightarrow \quad \frac{1}{k_{j+1}^{[p-2]-(p-3)}} < \frac{1}{(u-k_{j+1})_+^{[p-2]-(p-3)}}$$

$$\Rightarrow \quad \frac{(u-k_{j+1})_+^{[p-2]+3}}{[p-2]+3} k_{j+1}^{p-3-[p-2]} < \frac{1}{[p-2]+3} (u-k_{j+1})_+^p = \gamma(p) \, (u-k_{j+1})_+^p \leq \gamma(p) \, (u-k_j)_+^p$$

and we can then conclude that

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} p(p-1) \sum_{n=0}^{\infty} \binom{p-2}{n} k_{j+1}^{p-2-n} \frac{(u-k_{j+1})_+^{n+2}}{n+2} \varphi^{p-1}\varphi_t \, dxd\tau \tag{16}$$

$$\leq \gamma(p) \, 2^{jp} \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} (u-k_j)_+^p \, dxd\tau + C_\varphi \gamma(p) \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} (u-k_j)_+^p \, dxd\tau.$$

We can now consider the third term on the right - hand side of (5) with respect to level k_{j+1}. We have

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}} p(p-1)\sum_{n=0}^{\infty}\binom{p-2}{n}k_{j+1}^n\frac{(u-k_{j+1})_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau =$$

$$=\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}} p(p-1)\sum_{n=0}^{[p-2]+1}\binom{p-2}{n}k_{j+1}^n\frac{(u-k_{j+1})_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau+$$

$$+\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}} p(p-1)\sum_{n=[p-2]+2}^{\infty}\binom{p-2}{n}k_{j+1}^n\frac{(u-k_{j+1})_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau$$

$$\le\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}} p(p-1)\sum_{n=0}^{[p-2]+1}\binom{p-2}{n}k_{j+1}^n\frac{(u-k_{j+1})_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau.$$

Moreover

$$k_{j+1}<(u-k_{j+1})_+\quad\Rightarrow\quad k_{j+1}^n<(u-k_{j+1})_+^n,$$

and we obtain

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}} p(p-1)\sum_{n=0}^{\infty}\binom{p-2}{n}k_{j+1}^n\frac{(u-k_{j+1})_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau$$

$$\le C_\varphi\gamma(p)\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}} p(p-1)(u-k_{j+1})_+^p\,dxd\tau \tag{17}$$

$$\le C_\varphi\gamma(p)\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}} p(p-1)(u-k_j)_+^p\,dxd\tau.$$

As for the last term on the right - hand side of (5), it is easy to check that

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}=k_{j+1}\}}(u-k_{j+1})_+^p\varphi^{p-1}\varphi_t\,dxd\tau\le\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}=k_{j+1}\}}(u-k_j)_+^p\varphi^{p-1}\varphi_t\,dxd\tau. \tag{18}$$

Relying on (5) and (16) - (18) we conclude that

$$\frac{p-1}{p}\sup_{-\theta\rho^p<t<0}\int_{K_\rho}(u-k_{j+1})_+^p\varphi^p(x,t)\,dx+\iint_{Q(\rho,\theta\rho^p)}|D(u-k_{j+1})_+|^p\varphi^p\,dxd\tau$$

$$\le\gamma\left[\iint_{Q(\rho,\theta\rho^p)}(u-k_j)_+^p|D\varphi|^p\,dxd\tau+C_\varphi 2^{jp}\iint_{Q(\rho,\theta\rho^p)}(u-k_j)_+^p\,dxd\tau\right].$$

When $p=2$, this last inequality is the standard starting point to prove boundedness of u, as shown in [10], Chapter II, Section 6. In our case, even if we are dealing with a general $p>2$, it is not difficult to see that the same calculations still hold, due to the p-homogeneity of both sides.

Similar estimates are developed in [4], Chapter V, to obtain boundedness estimates for solutions of degenerate parabolic equations, like the parabolic p-laplacian ■.

4. - A Harnack Inequality

We can now come to the proof of Theorem 1. First of all let us recall the main Lemma of [12], which is actually a suitable adaptation to the parabolic setting of an idea introduced in [2] for the elliptic setting. We denote by $Q(\rho)$, $\rho>0$ any family of domains satisfying $Q(\rho)\subset Q(r)$ for $0<\rho<r$. We have

Proposition 5. *Let m, μ, c_0, δ be positive constants and let $w > 0$ be a measurable function defined in a neighborhood of $Q(1)$ and such that*

$$\sup_{Q(\rho)} w^p < \frac{c_0}{(r-\rho)^m} \iint_{Q(r)} w^p \, dx \tag{19}$$

for all ρ, r, p satisfying

$$\frac{1}{2} \leq \rho < r \leq 1, \qquad 0 < p < \mu^{-1}.$$

Moreover, let

$$|\{x \in Q(1) : \ \ln w > s\}| < \frac{c_0 \mu}{s^\delta} \tag{20}$$

for all $s > 0$. Then there exists a constant $\gamma = \gamma(\mu, m, c_0, \delta)$ such that

$$\sup_{Q(\frac{1}{2})} w < \gamma. \tag{21}$$

It is worth to notice that in [12] the parameter δ is taken equal to one, but as it is remarked in [3] any positive δ can do. In [12], Proposition 5 is the key point in proving the Harnack inequality for parabolic equations with bounded and measurable coefficients. Here we follow the same strategy and therefore we need the equivalent of (20) in our setting. We have

Proposition 6. *Fix $\theta > 0$ and $\sigma \in]0, 1[$. If u is a positive solution of (3) in $[(x_0, t_0) + Q(\rho, \theta\rho^p)]$, there exists a constant $c = c(u, \sigma)$ such that, for all $s > 0$*

$$|\{(x,t) \in Q_{\sigma\rho}{}^+ : \log u < -s - c\}| \leq \frac{C}{s^{p-1}} |Q(\rho, \theta\rho^p)| \tag{22}$$

and

$$|\{(x,t) \in Q_{\sigma\rho}{}^- : \log u > s - c\}| \leq \frac{C}{s^{p-1}} |Q(\rho, \theta\rho^p)| \tag{23}$$

where $Q_{\sigma\rho}{}^+ = [x_0 + K_{\sigma\rho}] \times]t_0 - \theta\sigma^p\rho^p, t_0]$ and $Q_{\sigma\rho}{}^- = [x_0 + K_{\sigma\rho}] \times]t_0 - \theta\rho^p, t_0 - \theta\sigma^p\rho^p]$. Here the constant C is independent of s, u, (x_0, t_0) and K_ρ.

Proof - Things are very much the same as in the proof of the analogous proposition of [12]. Here we closely follow the exposition given in Lemma 5.4.1 of [14]. First of all we set $(x_0, t_0) = (0, 0)$ as always and take $\theta = 1$. Let K'_ρ be any concentric ball larger than K_ρ. For any nonnegative $\zeta \in C_0^\infty(K'_\rho)$ we consider the test function $\varphi = \dfrac{\zeta^p}{u^{p-1}}$. If we insert it in (3), relying on the regularity of u we obtain

$$\int_{K'_\rho} \left[(u^{p-1})_t \frac{\zeta^p}{u^{p-1}} + |Du|^{p-2} Du \cdot D(\frac{\zeta^p}{u^{p-1}}) \right] dx = 0$$

and also

$$(p-1)\frac{\partial}{\partial t} \int_{K'_\rho} (\zeta^p \ln u) \, dx + (p-1) \int_{K'_\rho} \zeta^p \frac{1}{u^p} |Du|^p \, dx + p \int_{K'_\rho} |Du|^{p-2} \frac{\zeta^{p-1}}{u^{p-1}} Du \cdot D\zeta \, dx = 0.$$

If we set $w = -\log u$, we can rewrite the previous inequality as

$$\frac{\partial}{\partial t} \int_{K'_\rho} \zeta^p w \, dx = -\int_{K'_\rho} \zeta^p |Dw|^p \, dx + \frac{p}{p-1} \int_{K'_\rho} |Dw|^{p-2} \zeta^{p-1} Dw \cdot D\zeta \, dx$$

from which we obtain in a standard way

$$\frac{\partial}{\partial t} \int_{K'_\rho} \zeta^p w \, dx + C_1 \int_{K'_\rho} \zeta^p |Dw|^p \, dx \leq C_2 (\sup_{K'_\rho} |D\zeta|^p) \, |K'_\rho|. \tag{24}$$

Let us now choose $\zeta(x) = (1 - \frac{|x|}{\rho})_+$: ζ is not smooth, but it can easily be approximated by nonnegative C_0^∞ functions in K'_ρ. Then the weighted Poincaré inequality of Theorem 5.3.4 of [14] becomes

$$\int_{K_\rho} |w - W|^p \zeta^p \, dx \le A_0 \rho^p \int_{K_\rho} |Dw|^p \zeta^p \, dx \tag{25}$$

with

$$W = \frac{\int_{K_\rho} w \zeta^p \, dx}{\int_{K_\rho} \zeta^p \, dx}. \tag{26}$$

If we divide (24) by $\int_{K_\rho} \zeta^p \, dx$ and take into account (25) and (26), we obtain

$$\frac{\partial W}{\partial t} + \frac{1}{A_1 \rho^{N+p}} \int_{K_{\sigma\rho}} |w - W|^p \, dx \le \frac{A_2}{\rho^p}$$

for some constants A_1, $A_2 > 0$. We can rewrite this inequality as

$$\frac{\partial \bar{W}}{\partial t} + \frac{1}{A_1 \rho^{N+p}} \int_{K_{\sigma\rho}} |\bar{w} - \bar{W}|^p \, dx \le 0$$

where $\bar{w}(x,t) = w(x,t) - A_2 \rho^{-p}(t - t')$, $\bar{W}(t) = W(t) - A_2 \rho^{-p}(t - t')$ with $t' = -\sigma^p \rho^p$. We now set $c(u) = \bar{W}(t')$ and

$$K_t^+(s) = \{x \in K_{\sigma\rho} : \ \bar{w}(x,t) > c + s\},$$
$$K_t^-(s) = \{x \in K_{\sigma\rho} : \ \bar{w}(x,t) < c - s\}$$

and we can finish exactly as in Lemma 5.4.1 of [14], with the only difference that the exponent for s is $p - 1$ instead of 1 ▪.

We can now conclude with the

Proof of Theorem 1 - As always we assume $(x_0, t_0) = (0, 0)$. Fix $\theta > 0$ and let u be a positive solution of (3) in $K_{2\rho} \times] - 2^p \theta \rho^p, 0]$. By Proposition 6 with $\sigma = \frac{1}{2}$ we have

$$|\{(x,t) \in K_\rho \times] - 2\theta\rho^p, -\theta\rho^p] : \log u > s - c\}| \le$$

$$\le |\{(x,t) \in K_\rho \times] - 2^p\theta\rho^p, -\theta\rho^p] : \log u > s - c\}| \le \frac{C_1}{s^{p-1}} |Q(\rho, \theta\rho^p)|$$

and by Proposition 3

$$\sup_{[(0,-\theta\rho^p)+Q(\sigma\rho,\theta\sigma^p\rho^p)]} u^\epsilon \le \frac{C_2}{(1-\sigma)^{(N+p)}} \left(\fint\!\!\fint_{[(0,-\theta\rho^p)+Q(\rho,\theta\rho^p)]} u^\epsilon \, dx d\tau \right).$$

We can then apply Proposition 5 and conclude that

$$\sup_{[(0,-\theta\rho^p)+Q(\frac{\rho}{2},\frac{\theta}{2^p}\rho^p)]} e^c \, u \le C_3. \tag{27}$$

Analogously, by Proposition 6

$$|\{(x,t) \in K_\rho \times] - \theta\rho^p, 0] : \log u < -s - c\}| \le \frac{C_1}{s^{p-1}} |Q(\rho, \theta\rho^p)|$$

and by Proposition 4

$$\sup_{Q(\sigma\rho,\theta\sigma^p\rho^p)} \left(\frac{1}{u}\right)^\epsilon \le \frac{C_2}{(1-\sigma)^{(N+p)}} \left(\fint\!\!\fint_{Q(\rho,\theta\rho^p)} \left(\frac{1}{u}\right)^\epsilon dx d\tau \right).$$

We can then apply Proposition 5 and conclude that

$$\sup_{Q(\frac{\rho}{2},\frac{\theta}{2^p}\rho^p)} e^{-c}\, u^{-1} \leq C_4. \tag{28}$$

We can now multiply (27) by (28) and conclude that

$$\sup_{[(0,-\theta\rho^p)+Q(\frac{\rho}{2},\frac{\theta}{2^p}\rho^p)]} u \leq C_5 \inf_{Q(\frac{\rho}{2},\frac{\theta}{2^p}\rho^p)} u \tag{29}$$

with $C_5 = C_3 C_4$ ∎.

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Convergence of a Proximal-type Method for DC Functions

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Abstract: In this note we study the convergence of a descent-proximal method for finding critical points of a DC function. From a practical point of view, we propose an algorithmic pattern obtained by coupling the latter with cutting plane approximations.

AMS Subject Classification: Primary, 49J53, 65K10; Secondary, 90C25.
Key words. DC minimization, critical points, proximal point algorithm.

1 Introduction

The proximal point algorithm was introduced by Martinet for solving proper lower semi-continuous convex minimization problems and extensively studied by Rockafellar [4] in the context of monotone variational inequalities. It's will known that if we drop the convexity assumption on the objective function several problems arise. The proximal mapping is not well-defined and in general it is not nonexpansive anymore even in arbitrary small neighbourhoods of minima. Only few research has been proposed concerning the construction of solutions in this nonconvex case, see for instance [3]. Here we focus our attention on the method recently proposed by Sun et al. [5]. To find a critical point of $f := g-h$, it consists to increasing the function h along the direction of the subgradient and then decreasing the function f thanks to a proximal step. They proved that if the sequence generated by their algorithm is bounded, then every cluster point is a critical point of f. Here, we propose an elementary proof of their convergence result and we complete their study by proving the convergence of both the sequences of points, (x_k), and values, $(f(x_k))$. Afterwards, we provide conditions that ensure the boundedness of these sequences and discus the convergence of two approximate proximal schemes. Finally, suggest an algorithmic pattern obtained by coupling the latter method with cutting plane approximations.

Let $f := g - h$ where f and g are two convex lower semicontinuous and proper functions defined on $I\!R^n$. We consider the problem:

$$\min_{x \in I\!R^n} (g(x) - h(x)) \tag{1.1}$$

and the associated dual

$$\min_{y \in I\!R^n} (h^*(y) - g^*(y)) \tag{1.2}$$

$g^*(y) = \sup_{x \in I\!R^n}(g(x) - \langle y, x\rangle)$, h^* stand for the conjugate functions of g and h. It's well known that

$$\inf_{x \in I\!R^n} (g(x) - h(x)) = \inf_{y \in I\!R^n} (h^*(y) - g^*(y)),$$

and that a necessary condition for $x \in domf$ to be a local minimizer of f is $\partial h(x) \subset \partial g(x)$. As in general this necessary condition is hard to be reached, we will focus our attention on finding critical points of f, namely points satisfying the relaxed condition $\partial h(x) \cap \partial g(x) \neq \emptyset$.
Throughout the paper $f := g - h : I\!R^n \to \overline{I\!R}$ is a real DC function. We recall that the Moreau-Yosida approximate and the proximal mapping of g are defined for $c > 0$ by

$$g_c(x) := \inf_{u \in I\!R^n} \{g(u) + \frac{1}{2c}\|u - x\|^2\}$$

,

$$prox_{cg}(x) := (I + c\partial g(x))^{-1}(x) = \arg\min_u \{g(u) + \frac{1}{2c}\|u - x\|^2\},$$

and that a vector w is called a subgradient of g at $x \in domg$, if

$$g(u) \geq g(x) + \langle w, u - x\rangle \quad \forall u \in I\!R^n. \tag{1.3}$$

The set of all subgradients of g at x is denoted by $\partial g(x)$ and called the subdifferential of g at x. It worth mentioning the richness of the class of DC functions which contains de class of lower-$\mathcal{C}^2$ functions and constitutes a minimal realistic extension of the class of convex functions. It has been successfully used in many nonconvex applications such as finance, molecular biology, mutlicommodity network and seems particularly well suited to model several nonconvex industrial problems (Molecular optimization, computer's vision, fuel mixture ...).

2 Proximal Point Algorithm for DC functions

The method we will study is based on the following equivalence:

$$x \text{ is a critical point of } g - h \Leftrightarrow x = prox_{cg}(x + cw), \forall c > 0 \text{ and } w \in \partial h(x).$$

Thanks to this fixed-point formulation, Sun et al. [5] proposed an algorithm for finding a critical point of a DC function. This method combines proximal point algorithm with subgradient method, more precisely:

Algorithm: Proximal Method for DC Functions (PMDC)

Setp 1: Given $x_0, c_0 \geq c$. Set $k = 0$.
Step 2: Compute $w_k \in \partial h(x_k)$ and set $y_k = x_k + c_k w_k$.
Step 3: Compute $x_{k+1} = prox_{c_k g}(y_k)$ (Proximal step).
If $x_{k+1} = x_k$ stop. Otherwise increase k by 1 and loop to step 2.

The following proposition contains the convergence results of PMDC.

Proposition 2.1 *If $\underline{f} := \inf_{\mathbb{R}^n} f > -\infty$, then the sequence $(f(x_k))$ converges to some value $f_\infty \geq \underline{f}$ and the sequence (x_k) is asymptotically regular, namely*

$$\lim_{k\to+\infty}(g(x_k) - h(x_k)) = f_\infty \quad \textit{and} \quad \lim_{k\to+\infty} c_k^{-1}\|x_{k+1} - x_k\|^2 = 0. \tag{2.4}$$

Furthermore, we have

$$\lim_{k\to+\infty}(h^*(w_k) - g^*(c_k^{-1}(x_k - x_{k+1}) + w_k)) = f_\infty.$$

Moreover, if the sequences (x_k) and (w_k) are bounded then every cluster-point x_∞ (respectively w_∞) of (x_k) (respectively (w_k)) is a critical point of $g - h$ (respectively $h^* - g^*$).

Proof. Combining the fact that $c_k^{-1}(x_k - x_{k+1}) + w_k \in \partial g(x_{k+1})$, $w_k \in \partial h(x_k)$ and using the definition of the subdifferential, we obtain

$$f(x_{k+1}) \leq f(x_k) - c_k^{-1}\|x_k - x_{k+1}\|^2. \tag{2.5}$$

First, this shows that the Algorithm is a descent method. It is clear that the sequence $(f(x_k))$ decreases and since it is minorized, it converges to some f_∞. With this result in hand, we infer the asymptotical regularity of (x_k) from (2.5). Furthermore, since

$$h(x_k) + h^*(w_k) = \langle x_k, w_k\rangle$$

and

$$g(x_k) + g^*(c_k^{-1}(x_k - x_{k+1}) + w_k)) = \langle x_k, c_k^{-1}(x_k - x_{k+1}) + w_k)\rangle,$$

by subtracting the latter equalities, we obtain, at the limit, that

$$\lim_{k\to+\infty}(h^*(w_k) - g^*(c_k^{-1}(x_k - x_{k+1}) + w_k)) = f_\infty.$$

Now let (x_{k_ν}) and (w_{k_ν}) be two subsequences converging respectively to x_∞ and w_∞. By passing to the limit in the following relations

$$c_{k_\nu}^{-1}(x_{k_\nu} - x_{k_\nu+1}) + w_{k_\nu} \in \partial g(x_{k_\nu+1}) \text{ and } w_{k_\nu} \in \partial h(x_{k_\nu})$$

and taking into account the fact that the graphs of ∂g and ∂f are closed, we obtain

$$w_\infty \in \partial g(x_\infty) \quad \text{and} \quad w_\infty \in \partial h(x_\infty),$$

from which we infer that x_∞ is a critical point of $g - h$ and by duality that w_∞ is a critical point of $h^* - g^*$. ■

Remark 2.1 *It should be noticed that the assumption that (x_k) is bounded holds true if, for example, $g - h$ is coercive, namely $\lim_{\|x\|\to+\infty}(g(x) - h(x)) = +\infty$ and that (w_k) is bounded if, for instance, $x_k \in$ int (dom h) or h is a finite convex function.*

The following lemma will be needed in the proof of the next proposition.

Lemma 2.1 *(see [3]): Let (x_k) be a sequence of a compact metric space (E, d). If the set of its cluster points is finite and $\lim_{k\to\infty} d(x_k, x_{k+1}) = 0$, then the sequence (x_k) is convergent.*

We would like to emphasize that the set of cluster points of the sequence (x_k) is finite if, for example, h is polyhedral, namely $h = \tilde{h}(x) + \delta_C(x)$, where $\tilde{h}(x) = \max_{i=1\cdot\cdot m}\{\langle a_i, x\rangle - \alpha_i\}$ and C is a nonempty convex polyhedral subset of $I\!R^n$.

Proposition 2.2 *Suppose that*

1. *the sequence (x_k) is bounded and the set of its cluster points is finite,*
2. *there exists a bounded sequence (w_k) with $w_k \in \partial h(x_k)$.*

Then (x_k) converges to a critical point x_∞ of f and $\lim_{k\to+\infty} f(x_k) = f(x_\infty)$.

Proof. The first part of the proposition follows by invoking proposition 2.1 and lemma 2.1. Since $w_\infty \in \partial h(x_\infty)$ and $w_k \in \partial h(x_k)$, we have successively

$$h(x_k) \geq h(x_\infty) + \langle x_k - x_\infty, w_\infty\rangle \quad \forall k \in I\!N$$

and

$$h(x_\infty) \geq h(x_k) + \langle x_\infty - x_k, w_\infty\rangle \quad \forall k \in I\!N.$$

By passing to the limit and taking into account the fact that $\lim_{k\to+\infty} x_k = x_\infty$ and that the sequence (w_k) is bounded, we derive $\lim_{k\to+\infty} h(x_k) = h(x_\infty)$. On the other hand, since $c_k^{-1}(x_k - x_{k+1}) + w_k \in \partial g(x_{k+1})$, this implies

$$g(x_k) \leq g(x_\infty) + \langle c_k^{-1}(x_k - x_{k+1}) + w_k, x_k - x_\infty\rangle$$

which in turn ensues that $\limsup_{k\to+\infty} g(x_k) \leq g(x_\infty)$. By invoking the lower semi-continuity of g, we can write $g(x_\infty) \leq \liminf_{k\to+\infty} g(x_k)$ and the proof is complete. ■

In a practical point of view, we may consider errors in the computation of the iterates by replacing, for example, subdifferentials by their ε_k-enlargements (see, [2]). It is worth mentioning that the results are still valid for these type of approximate versions under a condition involving summability of the sequence $(\sqrt{c_k \varepsilon_k})_{k \in \mathbb{N}}$. Indeed, the definition of the ε-enlargement, $(\partial g)^{\varepsilon}(x)$, of the subdifferential of the proper convex lower-semicontinuous function g, namely

$$(\partial g)^{\varepsilon}(x) := \{v \in \mathbb{R}^n ; \langle u - v, y - x\rangle \geq -\varepsilon \quad \forall y, u \in \partial g(y)\} \text{ where } \varepsilon \geq 0,$$

directly yields that

$$\|(I + c_k(\partial g)^{\varepsilon_k})^{-1}(y_k) - prox_{c_k g}(y_k)\|^2 \leq c_k \varepsilon_k.$$

Another criterium, considered in [5], for an approximate calculation of the proximal mapping is provided by the following inexact minimization problem

$$x_{k+1} = \varepsilon_k - argmin\{g(x) + \frac{1}{2c_k}\|x - y_k\|^2\}, \tag{2.6}$$

in other words

$$c_k^{-1}(x_{k+1} - y_k) \in \partial_{\varepsilon_k} g(x_{k+1}).$$

Since the enlargement is larger than the approximate subdifferential, we can write

$$\partial_{\varepsilon_k} g(x_{k+1}) \subset (\partial g)^{\varepsilon_k}(x_{k+1}),$$

which leads to our approximate method. Thus convergence properties of (2.6) can be obtained as a consequence of those of our approximate algorithm.

To conclude, let us now focus our attention on the proximal step. It is will known that any convex function can be written as the envelope of its supporting hyperplanes, i.e.

$$g(x) = \max\{g(u) + \langle s(u), x - u\rangle, u \in \mathbb{R}^n, s(u) \in \partial g(u)\}.$$

Now assume that for any $x \in \mathbb{R}^n$, we can find $g(x)$ and a subgradient $s(x) \in \partial g(x)$ at x via an orcale (black box) (see, for example [1]). After k iterations of the PMDC, we have a sequence $(\hat{x}_i)_{i=0}^{k}$ which can be used to build the classical cutting plane approximation function $\hat{g}_k$ of g:

$$\hat{g}_k(x) := \max_{i=0}^{k}\{g(\hat{x}_i) + \langle s_i, x - \hat{x}_i\rangle\} \quad \text{with} \quad s_i := s(\hat{x}_i) \in \partial g(\hat{x}_i).$$

It results from the convexity that $\hat{g}_k$ is an under-estimate of g which is exact at each $\hat{x}_i$, namely

$$\hat{g}_k(x) \leq g(x),\ \hat{g}_k(\hat{x}_i) = g(\hat{x}_i) \quad \text{and} \quad s_i \in \partial\hat{g}_k(\hat{x}_i) \ \text{ for } i = 1 \cdot\cdot k.$$

The next approximation is better then the previous one, more precisely

$$\hat{g}_{k+1}(x) \geq \hat{g}_k(x) \ \text{ for all } x \in \mathbb{R}^n.$$

Coupling the proximal step with the bundle strategy yield to the following:

Algorithm: Bundle Proximal Method for DC Functions (BPMDC)

Step 1: Choose x_0, c_0 and set $k = 0, k_0 = 0, p = 1$.
Step 2: Call the orcale for x_k (to compute $w_k \in \partial h(x_k)$).
Compute $y_k = x_k + c_k w_k$ and $\hat{x}_k = \arg\min\{\hat{g}_k(x) + \frac{1}{2c_k}\|y_k - x\|^2\}$.
Step 3: If $g(\hat{x}_k) - \hat{g}_k(\hat{x}_k) \leq \varepsilon_p$, set $x_{k+1} := \hat{x}_k$, $k_p = k, p := p + 1$ (serious step). Otherwise set $x_{k+1} = x_k$ (null step).
Step 4: Call the orcale for $\hat{x}_k$ to obtain $s(\hat{x}_k)$ and update the cutting plane model $\hat{g}_{k+1}$ with $\hat{x}_k$, $g(\hat{x}_k)$ and $s(\hat{x}_k)$. Increase k by 1 and loop to step 2.

The stopping criterion is not specified, but we may use the stopping rule

$$g(\hat{x}_k) - \hat{g}(\hat{x}_k) \leq \delta \quad \text{for some prescribed tolerance} \quad \delta > 0.$$

For the choice of ε_p we refer, for instance, to [1]. We would like to emphasize that k_p denotes the iteration number of p^{th} serious step and that we have $\|\hat{x}_{k_p} - \tilde{x}_{k_p}\| \leq \sqrt{c_{k_p}\varepsilon_p}$ where $\tilde{x}_{k_p}$ is the the exact value of the proximal step corresponding to (x_{k_p}). Indeed, from step 2, we have $0 \in \partial\hat{g}_{k_p}(\hat{x}_{k_p}) + c_{k_p}^{-1}(\hat{x}_{k_p} - y_{k_p})$. Then for any $x \in I\!R^n$

$$\begin{aligned} g(x) \geq \hat{g}_{k_p}(x) \quad &\geq \quad \hat{g}_{k_p}(\hat{x}_{k_p}) + \langle c_{k_p}^{-1}(\hat{x}_{k_p} - y_{k_p}), x - \hat{x}_{k_p}\rangle \\ &\geq \quad g(\hat{x}_{k_p}) + \langle c_{k_p}^{-1}(\hat{x}_{k_p} - y_{k_p}), x - \hat{x}_{k_p}\rangle - \varepsilon_p, \end{aligned}$$

which implies that $c_{k_p}^{-1}(\hat{x}_{k_p} - y_{k_p}) \in \partial_{\varepsilon_p} g(\hat{x}_{k_p})$.
The convergence of the algorithm then follows provided that $\sum_p \sqrt{c_{k_p}\varepsilon_p} < +\infty$.

The usefulness of BPMDC to solve (1.1) depends on the availability of subroutines to solve the quadratic subproblems in step 2. This will be the case by considering the following equivalent quadratic problem with linear constraints:

$$\begin{array}{lrl} \min & r & + \dfrac{1}{2c_k}\|x - y_k\|^2 \\ & r & \geq g(x_i) + \langle s_i, x - x_i\rangle,\ i = 0, \cdots, k, \\ & (x, r) & \in I\!R^{n+1}, \end{array}$$

which is much easier and has a unique solution $(\hat{x}_k, \hat{r}_k)$ with $\hat{r}_k = \hat{g}_k(\hat{x}_k)$.

This note is aimed at studying the convergence properties of a proximal algorithm for minimizing DC functions in both exact and approximate forms and suggesting a concrete computational algorithm pattern for the proximal point.

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On Completely Generalized Nonlinear Random Variational-like Inclusions

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Abstract. In this paper, we consider random generalization of completely generalized nonlinear variational-like inclusions and propose an iterative algorithm for computing their approximate solutions. We prove that the approximate solutions obtained by proposed algorithm converge to the exact solution of our problem.
Key words: Completely generalized nonlinear variational-like inclusion, Monotone mapping, Algorithm, Convergence.
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1. INTRODUCTION AND FORMULATION

In the last two decades, the theory of variational inequalities was extended and generalized in many different directions because of its applications in mechanics, physics, optimization, economics and engineering sciences. The variational-like inequality also known as pre-variational inequality, is one of the generalised form of the variational inequality. The variational-like inequality problem and its generalizations are the powerful tool to study the nonconvex and nondifferentiable optimization problems; see for instant [1, 11, 15] and references therein.

The variational inequalities and quasi-variational inequalities for the random operators are called *random variational inequalities* and *random quasi-variational inequalities*; See for example [2, 3, 4, 6, 7, 9, 12, 14, 16] and references therein. Tan et al [13] studied random variational inequalities with applications to random minimization and nonlinear boundary problems, while Tarafdar and Yuan [14] gave the applications of random variational inequalities to random best approximation and fixed point theorems. In [9] and [16], random quasi-variational inequalities are studied with applications to random generalized games.

In this paper, we consider the completely generalized nonlinear variational-like inclusions for noncompact valued random mappings and suggest new iterative algorithm to compute the approximate solutions of our problem. We prove the existence of a random solution of our completely generalized nonlinear random variational-like inclusion and we study the convergence of random iterative sequences generated by the suggested algorithm.

Let (Ω, Σ) be a measurable space, where Ω is a set and Σ is a σ-algebra of subsets of Ω. Let H be a real separable Hilbert space whose norm and inner product are denoted by $\|.\|$ and $\langle ., . \rangle$, respectively. We denote by $\mathcal{B}(H)$ the class of Boral σ-field in H.

Definition 1. A mapping $x : \Omega \to H$ is said to be *measurable* if for any $B \in \mathcal{B}(H)$, $\{t \in \Omega : x(t) \in B\} \in \Sigma$.

Definition 2. A mapping $f : \Omega \times H \to H$ is called a *random operator* if for any $x \in H$, $f(t, x) = x(t)$ is measurable. A random operator f is said to be *continuous* if for any $t \in \Omega$, the mapping $f(t, .) : H \to H$ is continuous.

Definition 3. A multivalued map $T : \Omega \to 2^H$ is said to be *measurable* if for any $B \in \mathcal{B}(H)$, $T^{-1}(B) = \{t \in \Omega : T(t) \cap B \neq \emptyset\} \in \Sigma$.

Definition 4. A mapping $u : \Omega \to H$ is called a *measurable selection* of a multivalued measurable map $T : \Omega \to 2^H$ if u is measurable and for any $t \in \Omega$, $u(t) \in T(t)$.

Definition 5. A map $T : \Omega \to 2^H$ is called a *random multivalued map* if for any $x \in H$, $T(., x)$ is measurable. A random multivalued map $T : \Omega \times H \to CB(H)$ is said to be $\mathcal{H}$*-continuous* if for any $t \in \Omega$, $T(t, .)$ is continuous in the Hausdorff metric.

Given random multivalued maps $M, S, T : \Omega \times H \to 2^H$, random operators $f, g, p : \Omega \times H \to H$ with $Im(g) \cap dom\ \partial\phi \neq \emptyset$ and the random map $\eta : \Omega \times H \times H \to H$, we consider the following *completely generalized nonlinear random variational-like inclusion problem*:

$$\begin{cases} \text{Find measuable mappings } x,\ u,\ v,\ w : \Omega \to H \text{ such that} \\ \forall\ t \in \Omega \text{ and } y(t) \in H \\ x(t) \in H,\ u(t) \in M(t, x(t)),\ v(t) \in S(t, x(t)), \\ w(t) \in T(t, x(t)),\ g(t, u(t)) \cap dom\ \partial\phi \neq \emptyset \text{ and} \\ \langle p(t, u(t)) - (f(t, v(t)) - g(t, w(t))), \eta(t, y(t), x(t)) \rangle \geq \phi(x(t)) - \phi(y(t)), \end{cases}$$

where $\partial\phi$ is the subdifferential of a proper, convex and lower semicontinuous function $\phi : H \to \mathbb{R} \cup \{+\infty\}$. The set of measurable mappings (x, u, v, w) is called a *random solution* of completely generalized nonlinear random variational-like inclusion problem.

If $p \equiv 0$, f, g and M are identity maps, and S and T are single valued mappings, then completely generalized nonlinear random variational-like inclusion problem reduces to the problem of finding a measurable mapping $x : \Omega \to H$ such that $\forall\ t \in \Omega$ and $\forall\ y(t) \in H$,

$$\langle T(t, x(t)) - S(t, x(t)), \eta(t, y(t), x(t)) \rangle \geq 0. \tag{1}$$

Problem (1) is considered by Ding [3] in the setting of Banach spaces.

2. PRELIMINARIES

Lemma 1. [2] Let $T : \Omega \times H \to CB(H)$ be a $\mathcal{H}$-continuous random multivalued map. Then for any measurable mapping $w : \Omega \to H$, the multivalued map $T(., w(.)) : \Omega \to CB(H)$ is measurable.

Lemma 2. [2] Let $S, T : \Omega \to CB(H)$ be two measurable multivalued maps, $\epsilon > 0$ be constant and $v : \Omega \to H$ be a measurable selection of S. Then there exists a measurable selection $w : \Omega \to H$ of T such that $\forall\ t \in \Omega$,

$$\|v(t) - w(t)\| \leq (1 + \epsilon)\mathcal{H}(S(t), T(t)).$$

Definition 6. A random mapping $\eta : \Omega \times H \times H \to H$ is called:

(i) *monotone* if

$$\langle x(t) - y(t), \eta(t, x(t), y(t))\rangle \geq 0, \quad \forall\ x(t), y(t) \in H,\ t \in \Omega; \tag{2}$$

(ii) *strictly monotone* if the equality holds in (2) only when $x(t) = y(t)$;

(iii) *strongly monotone* if there exists a measurable function $q : \Omega \to (0, \infty)$ such that

$$\langle x(t) - y(t), \eta(t, x(t), y(t))\rangle \geq q(t)\|x(t) - y(t)\|^2, \quad \forall\ x(t), y(t) \in H,\ t \in \Omega;$$

(iv) *Lipschitz continuous* if there exists a measurable function $s : \Omega \to (0, \infty)$ such that

$$||\eta(t, x(t), y(t))|| \leq \delta\|x(t) - y(t)\|, \quad \forall\ x(t), y(t) \in H\ , t \in \Omega.$$

Definition 7. A random operator $g : \Omega \times H \to H$ is said to be *Lipschtiz continuous* if there exists a measurable function $r : \Omega \to (0, \infty)$ such that

$$||g(t, w_1(t)) - g(t, w_2(t))|| \leq r(t)\|w_1(t) - w_2(t)\|, \quad \forall\ w_1(t), w_2(t) \in H\ , t \in \Omega.$$

Definition 8. A random multivalued map $S : \Omega \times H \to CB(H)$ is said to be *$\mathcal{H}$-Lipschitz continuous* if there exists a measurable function $d : \Omega \to (0, \infty)$ such that

$$\mathcal{H}(S(t, x(t)), S(t, y(t))) \leq d(t)||x(t) - y(t)||, \quad \forall\ x(t), y(t) \in H\ t \in \Omega.$$

Definition 9. Let $f : H \to H$ be a random operator. A random multivalued map $S : H \to 2^H$ is said to be:

(i) *relaxed Lipschitz with respect to f* if there exists a measurable function $k : \Omega \to (0, \infty)$ such that
$\langle f(t, u(t)) - f(t, v(t)), x - y\rangle \leq -k(t)\|x(t) - y(t)\|^2,$
$\forall\ x(t), y(t) \in H,\ u(t) \in S(t, x(t)),\ v(t) \in S(t, y(t)),\ t \in \Omega;$

(ii) *relaxed monotone with respect to* f if there exists a measurable function $c : \Omega \to (0, \infty)$ such that
$\langle f(t, u(t)) - f(t, v(t)), x(t) - y(t)\rangle \geq -c(t)\|x(t) - y(t)\|^2,$
$\forall\ x(t), y(t) \in H,\ u(t) \in S(t, x(t)),\ v(t) \in S(t, y(t)),\ t \in \Omega.$

Definition 10. Let $\eta : \Omega \times H \times H \to H$ be a given random map. A random multivalued map $Q : \Omega \times H \to 2^H$ is called *η-monotone* if $\forall\ x(t), y(t) \in H$ and $t \in \Omega$,

$$\langle u(t) - v(t), \eta(t, x(t), y(t))\rangle \geq 0, \quad \forall\ u(t) \in Q(t, x(t)),\ v(t) \in Q(t, y(t)).$$

Q is called *maximal η-monotone* if and only if it is η-monotone and there is no other η-monotone random multivalued map whose graph strictly contains the graph of Q.

Assumption 1. The random operator $\eta : \Omega\times H\times H \to H$ satisfies the condition

$$\eta(t, y(t), x(t)) + \eta(t, x(t), y(t)) = 0, \quad \forall\ x(t), y(t) \in H,\ t \in \Omega.$$

Remark 1. If $\eta : \Omega \times H \times H \to H$ satisfies Assumption 1 and $\phi : H \to \mathbb{R}\cup\{+\infty\}$, then it is easy to see that the random multivalued map $\partial_\eta\phi : H \to 2^H$ is η-monotone.

The following result is the random version of Proposition due to Lee, Ansari and Yao [10].

Preposition 1. Let $\eta : \Omega \times H \times H \to H$ be strictly monotone random map and $Q : \Omega \times H \to 2^H$ an η-monotone random multivalued map. If, the range of $(I + \lambda Q)$, $R(I + \lambda Q) = H$, for $\lambda > 0$ and I is the identity operator, then Q is maximal η-monotone. Furthermore, the inverse operator $(I + \lambda Q)^{-1}$ is single valued.

3. AN ITERATIVE ALGORITHM

Throughout this section, we will assume that $\eta : \Omega \times H \times H \to H$ is strictly monotone and satisfies Assumption 1 and $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a functional such that $R(I + \lambda\partial_\eta\phi) = H$ for $\lambda > 0$.

From Proposition 1, we note that the mapping

$$J^\phi_\lambda(x(t)) = (I + \lambda\partial_\eta\phi)^{-1}(x(t)), \quad \forall\ x(t) \in H,\ t \in \Omega$$

is single valued.

Lemma 3. The set of measurable mappings $x,\ u,\ v,\ w : \Omega \to H$ is a random solution of completely generalized nonlinear random variational-like inclusion problem if and only if $\forall\ t \in \Omega,\ x(t) \in H,\ u(t) \in M(t, x(t)),\ v(t) \in S(t, x(t)),\ w(t) \in T(t, x(t))$ satisfy the following relation:

$$x(t) = J^\phi_{\lambda(t)}[x(t) - \lambda(t)(p(t, u(t)) - (f(t, v(t)) - g(t, w(t)))], \tag{3}$$

where $\lambda : \Omega \to (0, \infty)$ is a measurable mapping and $J^{\phi}_{\lambda(t)} = [I + \lambda(t)\partial\phi]^{-1}$ is so called proximal map on H and I stands for the identity operator on H.

Proof. From the definition of $J^{\phi}_{\lambda(t)}$, it follows that

$$x(t) - \lambda(t)[p(t, u(t)) - (f(t, v(t)) - g(t, w(t)))] \in x(t) + \lambda(t)\partial_{\eta}\phi x(t)$$

and hence

$$[f(t, v(t)) - g(t, w(t))] - p(t, u(t)) \in \lambda(t)\partial_{\eta}\phi x(t).$$

By using the definition of η-subdifferential, we have

$$\langle (f(t, v(t)) - g(t, w(t))) - p(t, u(t)), \eta(t, y(t), x(t)) \rangle \leq \phi(y(t)) - \phi(x(t)) \ \forall \ y(t) \in H, \ t \in \Omega.$$

Thus (x, u, v, w) is a random solution of completely generalized nonlinear random variational-like inclusion problem.

To obtain the approximate solutions of completely generalized nonlinear random variational-like inclusion problem we can apply a successive approximation method to the problem of solving

$$x(t) \in Q(t, x(t)) \tag{4}$$

for all $t \in \Omega$, where

$$Q(t, x(t)) = x(t) + J^{\phi}_{\lambda(t)}[x(t) - \lambda(t)((p(t, u(t)) - (f(t, S(t, v(t))) - g(t, T(t, w(t))))].$$

Based on (3) and (4), we propose the following algorithm to compute the approximate solutions of completely generalized nonlinear random variational-like inclusion problem.

Algorithm 1. Let $M, S, T : \Omega \times H \to CB(H)$ be $\mathcal{H}$-continuous random multivalued maps and $f, g, p : \Omega \times H \to H$ be continuous random operators. For any given measurable mapping $x_0 : \Omega \to H$, the multivalued mappings $M(., x_0(.)),\ S(., x_0(.)),\ T(., x_0(.)) : \Omega \to CB(H)$ are measurable by Lemma 1. Hence there exist measurable selection $u_0 : \Omega \to H$ of $M(., x_0(.))$, measurable selection $v_0 : \Omega \to H$ of $S(., x_0(.))$ and measurable selection $w_0 : \Omega \to H$ of $T(., x_0(.))$ by Himmelberg [5]. Let

$$x_1(t) = x_0(t) + J^{\phi_0}_{\lambda(t)}[x_0(t) - \lambda(t)((p(t, u_0(t)) - (f(t, v_0(t))) - g(t, w_0(t)))].$$

It is easy to see that $x_1 : \Omega \to H$ is measurable. By Lemma 2, there exist measurable selections $u_1 : \Omega \to H$ of $M(., x_1(.))$, measurable selection $v_1 : \Omega \to H$ of $S(., x_1(.))$ and measurable selection $w_1 : \Omega \to H$ of $T(., x_1(.))$ such that $\forall \ t \in \Omega$,

$$\begin{aligned}
\|u_0(t) - u_1(t)\| &\leq (1 + 1)\ \mathcal{H}(M(t, x_0(t)), M(t, x_1(t))), \\
\|v_0(t) - v_1(t)\| &\leq (1 + 1)\ \mathcal{H}(S(t, x_0(t)), S(t, x_1(t))), \\
\|w_0(t) - w_1(t)\| &\leq (1 + 1)\ \mathcal{H}(T(t, x_0(t)), T(t, x_1(t))).
\end{aligned}$$

Let

$$x_2(t) = x_1(t) + J^{\phi_1}_{\lambda(t)}[x_1(t) - \lambda(t)((p(t,u_1(t)) - (f(t,v_1(t))) - g(t,w_1(t)))],$$

then $x_2 : \Omega \to H$ is measurable.

By induction, we can obtain sequences $\{x_n(t)\}, \{u_n(t)\}, \{v_n(t)\}$ and $\{w_n(t)\}$ as follows:

$$x_{n+1}(t) = x_n(t) + J^{\phi_n}_{\lambda(t)}[x_n(t) - \lambda(t)((p(t,u_n(t)) - (f(t,v_n(t))) - g(t,w_n(t)))], \quad (5)$$

$$\|u_n(t) - u_{n+1}(t)\| \leq (1 + (n+1)^{-1})\ \mathcal{H}(M(t,x_n(t)), M(t,x_{n+1}(t))),$$
$$\|v_n(t) - v_{n+1}(t)\| \leq (1 + (n+1)^{-1})\ \mathcal{H}(S(t,x_n(t)), S(t,x_{n+1}(t))),$$
$$\|w_n(t) - w_{n+1}(t)\| \leq (1 + (n+1)^{-1})\ \mathcal{H}(T(t,x_n(t)), T(t,x_{n+1}(t))),$$

for any $t \in \Omega$ and $n = 0, 1, 2, \ldots$.

Lemma 4. Let $\eta : \Omega \times H \times H \to H$ be a strongly monotone and Lipschitz continuous random map with constants $q(t) > 0$ and $s(t) > 0$, respectively , and satisfy Assumption 1. Then

$$\|J^{\phi}_{\lambda(t)}x(t) - J^{\phi}_{\lambda(t)}y(t)\| \leq \tau(t)\|x(t) - y(t)\| \quad \forall\ x(t), y(t) \in H,$$

where $\tau(t) = \frac{s(t)}{q(t)}$.

Proof. For the proof see Lemma 3 of [10].

Theorem 1. *Let $\eta : \Omega \times H \times H \to H$ be strongly monotone and Lipschitz continuous random map with constants $q(t) > 0$ and $s(t) > 0$, respectively, and satisfy Assumption 1. Let $f, g, p : \Omega \times H \to H$ be Lipschitz continuous with corresponding constants $\xi(t), r(t)$ and $\sigma(t)$, respectively. Let $M, S, T : \Omega \times H \to CB(H)$ be $\mathcal{H}$ -Lipschtiz continuous with corresponding constants $\gamma(t), h(t)$ and $d(t)$, respectively and S be the relaxed Lipschtiz with respect to f with constant $k(t)$ and T be relaxed monotone with respect to g with constant $c(t)$. For each n, let $\phi_n : H \to \mathbb{R} \cup \{+\infty\}$ and $\phi : H \to \mathbb{R} \cup \{+\infty\}$ be mappings such that $R(I + \lambda(t)\partial_n \phi_n) = R(I + \lambda(t)\partial_n \phi) = H$ for $\lambda(t) > 0$. Assume that*

$$\lim_{n \to +\infty} \|J^{\phi_n}_{\lambda(t)}z(t) - J^{\phi_{n-1}}_{\lambda(t)}z(t)\| = 0, \quad \forall\ z(t) \in H$$

and if

$$\left|\lambda(t) - \frac{k(t) - c(t)}{[\xi(t)h(t) + r(t)d(t)]^2 - \sigma^2(t)\gamma^2(t)}\right|$$
$$< \frac{\sqrt{(c(t) - k(t))^2 - [(\xi(t)h(t) + r(t)d(t))^2 - \sigma^2(t)\gamma^2(t)]}}{[\xi(t)h(t) + r(t)d(t)]^2 - \sigma^2(t)\gamma^2(t)}$$
$$c(t) - k(t) > \sqrt{[\xi(t)h(t) + \gamma(t)d(t)]^2 - \sigma^2(t)\gamma^2(t)} \quad (6)$$

$$\xi(t)h(t) + r(t)d(t) > \sigma(t)\gamma(t).$$

Then there exists a set of elements $x(t) \in H, u(t) \in M(t, x(t)), v(t) \in S(t, x(t))$ *and* $w(t) \in T(t, x(t))$ *which is a solution of completely generalized nonlinear random variational-like inclusion problem and* $x_n(t) \to x(t), u_n(t) \to u(t), v_n(t) \to v(t), w_n(t) \to w(t)$ *as* $n \to \infty$, *where* $\{x_n(t)\}, \{u_n(t)\}, \{v_n(t)\}$ *and* $\{w_n(t)\}$ *are the random sequences obtained by Algorithm 1.*

Proof. From (3), we have

$$\|x_{n+1}(t) - x_n(t)\| = \|x_n(t) - x_{n-1}(t) + J^{\phi_n}_{\lambda(t)}(h(x_n(t)) - J^{\phi_{n-1}}_{\lambda(t)}(h(x_{n-1}(t)))\|, \quad (7)$$

where

$$h(x_n(t)) = x_n(t) - \lambda(t)[(p(t, u_n(t)) - (f(t, v_n(t)) - g(t, w_n(t)))].$$

By introducing the term $J^{\phi_n}_{\lambda(t)}(h(x_{n-1}(t))$, we get

$$\|J^{\phi_n}_{\lambda}(h(x_n(t))) - J^{\phi_{n-1}}_{\lambda}(h(x_{n-1}(t)))\|$$
$$\leq \|J^{\phi_n}_{\lambda}(h(x_n(t))) - J^{\phi_n}_{\lambda}(h(x_{n-1}(t)))\| + \|J^{\phi_n}_{\lambda}(h(x_{n-1}(t))) - J^{\phi_{n-1}}_{\lambda}(h(x_{n-1}(t)))\|. \quad (8)$$

By Lemma (4), we have

$$\|J^{\phi_n}_{\lambda}(h(x_n(t))) - J^{\phi_{n-1}}_{\lambda}(h(x_{n-1}(t)))\|$$
$$\leq \tau(t)\|h(x_n(t)) - h(x_{n-1}(t))\| + \|J^{\phi_n}_{\lambda}(h(x_{n-1}(t))) - J^{\phi_{n-1}}_{\lambda}(h(x_{n-1}(t)))\|, \quad (9)$$

where $\tau(t) = \frac{s(t)}{q(t)}$, and

$$\begin{aligned} \|h(x_n(t)) - h(x_{n-1}(t))\| &= \|x_n(t) - \lambda(t)(p(t, u_n(t)) - (f(t, v_n(t)) \\ &\quad -g(t, w_n(t)))) - x_{n-1}(t) + \lambda(t)(p(t, u_{n-1}(t)) \\ &\quad -(f(t, v_{n-1}(t)) - g(t, w_{n-1}(t)))\| \\ &\leq \|x_n(t) - x_{n-1}(t) + \lambda(t)(f(t, v_n(t) - f(t, v_{n-1}(t)) \\ &\quad -\lambda(t)(g(t, w_n(t) - g(t, w_{n-1}(t))\| \\ &\quad +\lambda(t)\tau(t)\|(p(t, u_n(t) - p(t, u_{n-1}(t))\|. \quad (10) \end{aligned}$$

From (7) - (10), we get

$$\begin{aligned} \|x_{n+1}(t) - x_n(t)\| &\leq \|x_n(t) - x_{n-1}(t)\| \\ &\quad +\tau(t)\|x_n(t) - x_{n-1}(t) + \lambda(t)(f(t, v_n(t) \\ &\quad -f(t, v_{n-1}(t)) - \lambda(t)(g(t, w_n(t) - g(t, w_{n-1}(t))\| \\ &\quad +\lambda(t)\tau(t)\|p(t, u_n(t) - p(t, u_{n-1}(t))\| \\ &\quad +\|J^{\phi_n}_{\lambda(t)}(h(x_{n-1}(t)) - J^{\phi_n}_{\lambda(t)}(h(x_{n-1}(t))\|. \quad (11) \end{aligned}$$

Since M, S and T are $\mathcal{H}$- Lipschitz continuous, and f, g and p are Lipschtiz continuous, we have

$$\begin{aligned} \|p(t, u_n(t)) - p(t, u_{n-1}(t))\| &\leq \sigma(t)\|u_n(t) - u_{n-1}(t)\| \\ &\leq \sigma(t)\gamma(t)(1 + 1/n)\|x_n(t) - x_{n-1}(t)\| \quad (12) \end{aligned}$$

$$\begin{aligned}\|f(t,v_n(t))-f(t,v_{n-1}(t))\| &\leq \xi(t)\|v_n(t)-v_{n-1}(t)\| \\ &\leq \xi(t)h(t)(1+1/n)\|x_n(t)-x_{n-1}(t)\| \qquad (13)\end{aligned}$$

$$\begin{aligned}\|g(t,w_n(t))-g(t,w_{n-1}(t))\| &\leq r(t)\|w_n(t)-w_{n-1}(t)\| \\ &\leq r(t)d(t)(1+1/n)\|x_n(t)-x_{n-1}(t)\|. \qquad (14)\end{aligned}$$

Further , since S is relaxed Lipschitz and T is relaxed monotone , we have

$$\|x_n(t)-x_{n-1}(t)+\lambda(t)(f(t,v_n(t)-f(t,v_{n-1}(t))-\lambda(t)(g(t,w_n(t)-g(t,w_{n-1}(t))\|^2$$

$$\begin{aligned}&= \|x_n(t)-x_{n-1}(t)\|^2+2\lambda(t)\langle f(t,v_n(t))-f(t,v_{n-1}(t)),x_n(t)\\ &\quad -x_{n-1}(t)\rangle-2\lambda(t)\langle g(t,w_n(t))-g(t,w_{n-1}(t)),x_n(t)-x_{n-1}(t)\rangle\\ &\quad +\lambda^2(t)\|f(t,v_n(t))-f(t,v_{n-1}(t))-(g(t,w_n(t))-g(t,w_{n-1}(t))\|^2\\ &\leq [1-2\lambda(t)(k(t)-c(t))+\lambda^2(t)(1+1/n)^2(\xi(t)h(t)\\ &\quad +r(t)d(t))^2]\|x_n(t)-x_{n-1}(t)\|^2. \qquad (15)\end{aligned}$$

From (11) - (15) , it follows that

$$\begin{aligned}\|x_{n+1}(t)-x_n(t)\| &= \theta_n\|x_n(t)-x_{n-1}(t)\|\\ &\quad +\|J^{\phi_n}_{\lambda(t)}(h(x_{n-1}(t))-J^{\phi_{n-1}}_{\lambda(t)}(h(x_{n-1}(t)))\|, \qquad (16)\end{aligned}$$

where

$$\begin{aligned}\theta_n(t) &= 1+\tau(t)\sqrt{(1-2\lambda(t))(k(t)-c(t))+\lambda^2(t)(1+1/n)^2[\xi(t)h(t)+r(t)d(t)]^2}\\ &\quad +\lambda(t)\sigma(t)\gamma(t)\tau(t)(1+1/n).\end{aligned}$$

Let,

$$\begin{aligned}\theta(t) &= 1+\tau(t)\sqrt{(1-2\lambda(t))(k(t)-c(t))+\lambda^2(t)[\xi(t)h(t)+r(t)d(t)]^2}\\ &\quad +\lambda(t)\sigma(t)\gamma(t)\tau(t).\end{aligned}$$

Then $\theta_n(t)\to\theta(t)$ as $n\to\infty$. It follows from (6) that $\theta(t)<1$.

Since $\lim_{n\to+\infty}\|J^{\phi_n}_{\lambda(t)}z(t)-J^{\phi_{n-1}}_{\lambda(t)}z(t)\|=0$, It follows from (16) that $\{x_n(t)\}$ is a Cauchy sequence in H. Since H is complete, then there exists a measurable map $x:\Omega\to H$ such that $x_n(t)\to x(t)$, for all $t\in\Omega$. Now we prove that $u_n(t)\to u(t)\in M(t,x(t)), v_n(t)\to v(t)\in S(t,x(t))$ and $w_n(t)\to w(t)\in T(t,x(t))$. In fact , It follows from Algorithm 1 that

$$\|u_n(t)-u_{n-1}(t)\|\leq(1+1/n)\gamma(t)\|x_n(t)-x_{n-1}(t)\|,$$

$$\|v_n(t)-v_{n-1}(t)\|\leq(1+1/n)h(t)\|x_n(t)-x_{n-1}(t)\|,$$

$$\|w_n(t)-w_{n-1}(t)\|\leq(1+1/n)d(t)\|x_n(t)-x_{n-1}(t)\|,$$

which implies that $\{u_n(t)\}, \{v_n(t)\}$ and $\{w_n(t)\}$ are also Cauchy sequences in H. Let $u_n(t) \to u(t), v_n(t) \to v(t), w_n(t) \to w(t)$ as $n \to \infty$. We have

$$\begin{aligned} d(v(t), S(t, x(t)) &= inf\ \{\|v(t) - y(t)\| : y \in S(t, x(t)\} \\ &\leq \|v(t) - v_n(t)\| + d(v_n(t), S(t, x(t)) \\ &\leq \|v(t) - v_n(t)\| + \mathcal{H}(S(t, x_n(t)), S(t, x(t)) \\ &\leq \|v(t) - v_n(t)\| + h\|x_n(t) - x(t)\| \to 0 \text{ as } n \to \infty \end{aligned}$$

Hence $v(t) \in S(t, x(t))$. Similarly we can prove that $u(t) \in M(t, x(t)), w(t) \in T(t, x(t))$. This complete the proof.

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Mild solutions for impulsive semilinear evolution differential inclusions

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Abstract

In this paper we prove a sufficient condition for the existence of mild solutions for an impulsive Cauchy problem monitored by the semilinear evolution differential inclusion $x'(t) \in A(t)x(t) + F(t, x(t))$ where $\{A(t)\}_{t\in[0,b]}$ is a family of linear operators in a Banach space E generating an evolution operator and F is a Carathèodory type multifunction. Since we do not assume any hypothesis on the impulse functions, our existence theorem extends in a broad sense a proposition obtained by Benchohra, Henderson and Ntouyas.

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Key words and Phrases: *impulsive semilinear evolution differential inclusion, evolution system, generalized Cauchy operator, mild solution, measure of noncompactness, upper semi-continuous multifunction.*

1 Introduction

The impulsive Cauchy problems have advantage over the traditional Cauchy problems since they can be used to model processes which are subjected to abrupt changes. In the mathematical simulation it is convenient to assume that these changes take place momentarily and that the change of the state is given by a jump.

During recent years, the impulsive differential equations have been an object of intensive investigation because of the wide possibilities for their application in various fields of science and technology as theoretical physics, population dynamics, economics, etc. (see, e.g. [6], [15], [21], [24]).

Several monographs related to this subject have been published (see, e.g. Bainov and Covachev [2], Bainov, Lakshmikantham and Simeonov [3], Bainov and Simeonov [4], [5], Mil'man and Myshkis [20], Samoilenko and Perestyuk [23]).

Later Liu [18], by means of the semigroup theory, has proved the existence and uniqueness of mild solutions for an impulsive Cauchy problem (with Lipschitz impulse functions) controlled by an evolution equation

$$x'(t) = Ax(t) + f(t, x(t)) \ ,$$

being A an unbounded operator generating a strongly continuous semigroup and f a continuous function, Lipschitz with respect to the second variable.

In last years some works concerning impulsive Cauchy problems have appeared also monitored by semilinear differential inclusions (see, e.g. [1], [7], [8], [9], [12], [13]).

In this note we provide a sufficient condition for the existence of *mild solutions* (see Definition 1) for the following impulsive Cauchy problem governed by a semilinear evolution differential inclusion

$$(P)\begin{cases} x'(t) \in A(t)x(t) + F(t, x(t)) & , \text{a.e. } t \in [0,b]\,,\ t \neq t_k\,,\ k = 1, \cdots, m \\ \\ x(t_k^+) = x(t_k) + I_k(x(t_k)) & , k = 1, \cdots, m \\ \\ x(0) = a \in E \end{cases}$$

where $\{A(t)\}_{t\in[0,d]}$ is a family of linear operators (not necessarily bounded or closed) in the Banach space E generating an evolution operator; F is a Carathèodory type multifunction; $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$; $I_k : E \to E$, $k = 1, \cdots, m$, are impulse functions and $x(t^+) = \lim_{s\to t^+} x(s)$.

We achieve our result by applying some of the theorems stated in [10] and a fixed point theorem for condensing multifunctions (see [16], Corollary 3.3.1).

We observe that our hypotheses are more general than conditions assumed by Liu in [18].

Moreover, we remark that our existence theorem extends in a broad sense an analogous proposition obtained by Benchohra, Henderson and Ntouyas in [7]. In fact, they require the impulse functions to be continuous and to satisfy a suitable property, whereas we don't

assume any hypothesis on these.

2 Preliminaries

Let X and Y be topological spaces and let us denote by $P(Y)$ the collection of all nonempty subsets of Y.

A multivalued map (multimap) $\mathcal{F} : X \to P(Y)$ is said to be:

upper semicontinuous (u.s.c) if $\mathcal{F}^{-1}(V) = \{x \in X \ : \ \mathcal{F}(x) \subset V\}$ is an open subset of X for every open $V \subset Y$;

closed if its graph $\Gamma_F = \{(x, y) : y \in F(x)\}$ is a closed subset of the space $X \times Y$.

Let $(E, \|\cdot\|)$ be a Banach space and $(\mathcal{A}, \geq)$ be a (partially) ordered set.

We recall (see, e.g. [16]) that a function $\beta : P(E) \to \mathcal{A}$ is called *a measure of noncompactness* (MNC) in E if

$$\beta(\overline{co}\Omega) = \beta(\Omega)$$

for every $\Omega \in P(E)$.

Moreover, a MNC is said to be:

monotone if $\Omega_0, \Omega_1 \in P(E)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;

nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E$, $\Omega \in P(E)$;

real if $\mathcal{A} = [0, +\infty]$ with the natural ordering and $\beta(\Omega) < +\infty$ for every bounded Ω.

If $\mathcal{A}$ is a cone in a Banach space we say that the MNC β is *regular* if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

In the following we will often use the *Hausdorff MNC*

$$\chi(\Omega) = \inf\{\varepsilon > 0 \ : \ \Omega \text{ has a finite } \varepsilon\text{-net}\}$$

which is a MNC possessing all the properties described above.

Let W be a closed subset of a Banach space E, $\beta : P(E) \to \mathcal{A}$ be a MNC on E and $K(E)$ $[Kv(E)]$ denote the collection of all nonempty compact [compact convex] subsets of E.

A multimap $\mathcal{F} : W \to K(E)$ is said to be *β-condensing* if for every $\Omega \subset W$ the relation

$$\beta(\mathcal{F}(\Omega)) \geq \beta(\Omega)$$

implies the relative compactness of Ω.

Also the following notions from the literature (see, e.g. [11], [14], [16]) will be used throughout this paper.

Let J and $\bar{J}$ be respectively a bounded interval of the real line and its closure.

A multifunction $\mathcal{G} : \bar{J} \to K(E)$ is *strongly measurable* if there exists a sequence $(\mathcal{G}_n)_{n=1}^{\infty}$ of step multifunctions such that

$$H(\mathcal{G}_n(t), \mathcal{G}(t)) \to 0$$

as $n \to \infty$ for a.e. $t \in \bar{J}$ (on the interval $\bar{J}$ we consider the Lebesgue measure and H is the Hausdorff metric on $K(E)$).

Every strongly measurable multifunction $\mathcal{G}$ admits *a strongly measurable selection* $g : \bar{J} \to E$, i.e. g is strongly measurable and $g(t) \in \mathcal{G}(t)$ for a.e. $t \in \bar{J}$.

Let the symbol $L^1(\bar{J}; E)$ denote the space of all Bochner summable functions; moreover, for the sake of simplicity, the symbol $L^1_+(\bar{J})$ will denote the space $L^1(\bar{J}; I\!\!R^+)$.

A multifunction $\mathcal{G} : \bar{J} \to K(E)$ is

integrable if it has a summable selection $g \in L^1(\bar{J}; E)$;

integrably bounded if there exists a summable function $\omega(\cdot) \in L^1_+(\bar{J})$ such that

$$\|\mathcal{G}(t)\| := \sup\{\|g\| \ : \ g \in \mathcal{G}(t)\} \leq \omega(t), \qquad \text{a.e. } t \in \bar{J}.$$

The set of all summable selections of the multifunction $\mathcal{G}$ on the interval $\bar{J}$ will be denoted by $\mathcal{S}^1_{\mathcal{G},\bar{J}}$.

A countable set $\{f_n\}_{n=1}^{\infty} \subset L^1(\bar{J}; E)$ is said to be *semicompact* if:

i) it is integrably bounded: $\|f_n(t)\| \leq \omega(t)$ for a.e. $t \in \bar{J}$ and every $n \geq 1$ where $\omega(\cdot) \in L^1_+(\bar{J})$

ii) the set $\{f_n(t)\}_{n=1}^{\infty}$ is relatively compact for a.e. $t \in \bar{J}$.

By the symbol $C(\bar{J}, E)$ we will denote the set of all the continuous functions defined on $\bar{J}$ which take values in E.

Let $[0, b]$, $b > 0$, be a fixed interval of the real line and $\Delta = \{(t, s) \in [0, b] \times [0, b] \ : \ 0 \leq s \leq t \leq b\}$.

Now we recall a basic definition (see, e.g. [22]).

A two parameter family $\{T(t, s)\}_{(t,s)\in\Delta}$, $T(t, s) : E \to E$ bounded linear operator, $(t, s) \in \Delta$, is called an *evolution system* if the following conditions are satisfied:

1. $T(s, s) = I, \quad T(t, r)T(r, s) = T(t, s)$ for $0 \leq s \leq r \leq t \leq b$;

2. $(t, s) \mapsto T(t, s)$ is strongly continuous on Δ (see, e.g. [17]).

For any evolution system, we can consider the respective *evolution operator* $T : \Delta \to \mathcal{L}(E)$, where $\mathcal{L}(E)$ is the space of all bounded linear operators in E.

In the sequel we will also need the following definition introduced in [10].

Given an evolution operator $T : \Delta \to \mathcal{L}(E)$, the operator $G : L^1([0,h];E) \to C([0,h];E)$ $(0 < h \le b)$ defined by

$$Gf(t) = \int_0^t T(t,s)f(s)\,ds \quad t \in [0,h]$$

is said to be the *generalized Cauchy operator*.

In order to prove our existence result we will use the following propositions.

Lemma 1. ([10], Theorem 2) *The generalized Cauchy operator G satisfies properties*

$(G1)$ *there exists* $\zeta \ge 0$ *such that*

$$\|Gf(t) - Gg(t)\| \le \zeta \int_0^t \|f(s) - g(s)\| ds$$

for every $f, g \in L^1([0,h];E)$, $0 \le t \le h$;

$(G2)$ *for any compact* $K \subset E$ *and sequence* $(f_n)_{n=1}^\infty$, $f_n \in L^1([0,h];E)$, *such that* $\{f_n(t)\}_{n=1}^\infty \subset K$ *for a.e.* $t \in [0,h]$, *the weak convergence* $f_n \rightharpoonup f_0$ *implies the convergence* $Gf_n \to Gf_0$.

Lemma 2. ([16], Theorem 5.1.1) *Let* $S : L^1([0,h];E) \to C([0,h];E)$ *be an operator satisfying condition* $(G2)$ *and the Lipschitz condition (weaker than* $(G1)$*)*

$(G1')$ $\|Sf - Sg\|_C \le \zeta \|f - g\|_{L^1([0,h];E)}$ *(where $\|\cdot\|_C$ is the usual sup-norm).*

Then for every semicompact set $\{f_n\}_{n=1}^{\infty} \subset L^1([0,h];E)$ *the set* $\{Sf_n\}_{n=1}^{\infty}$ *is relatively compact in* $C([0,h];E)$ *and, moreover, if* $f_n \rightharpoonup f_0$ *then* $Sf_n \to Sf_0$.

Lemma 3. ([16], Theorem 4.2.2) *Let the operator* $S : L^1([0,h];E) \to C([0,h];E)$ *satisfy conditions* $(G1)$ *and* $(G2)$ *and let the set* $\{f_n\}_{n=1}^{\infty}$ *be integrably bounded with the property* $\chi(\{f_n(t)\}_{n=1}^{\infty}) \leq \eta(t)$ *for a.e.* $t \in [0,h]$ *where* $\eta(\cdot) \in L^1_+([0,h])$ *and* χ *is the Hausdorff MNC. Then*

$$\chi(\{Sf_n(t)\}_{n=1}^{\infty}) \leq 2\zeta \int_0^t \eta(s)ds$$

for all $t \in [0,h]$, *where* $\zeta \geq 0$ *is the constant in condition* $(G1)$.

3 Existence Theorem

In this section we prove the existence of mild solutions for the impulsive Cauchy problem (P).

We will assume the following hypothesis on the linear part of the inclusion:

(A) $\{A(t)\}_{t\in[0,b]}$ is a family of linear (not necessarily bounded) operators, $A(t) : D(A) \subset E \to E$, $D(A)$ not depending on t and dense subset of E, $t \in [0,b]$, generating an evolution operator $T : \Delta \to \mathcal{L}(E)$, i.e. there exists an evolution system $\{T(t,s)\}_{(t,s)\in\Delta}$ such that, on the region $D(A)$, each operator $T(t,s)$ is strongly differentiable relative to t and s, while

$$\frac{\partial T(t,s)}{\partial t} = A(t)T(t,s) \qquad \text{and} \qquad \frac{\partial T(t,s)}{\partial s} = -T(t,s)A(s) \quad , \quad (t,s) \in \Delta$$

(for more details see, e.g. [17], [19]).

$F : [0,b] \times E \to Kv(E)$ is a multivalued nonlinearity such that

(F) $(F1)$ for every $x \in E$ the multifunction $F(\cdot, x) : [0, b] \to Kv(E)$ admits a strongly measurable selector;

$(F2)$ for a.e. $t \in [0, b]$ the multifunction $F(t, \cdot) : E \to Kv(E)$ is u.s.c.;

$(F3)$ there exists a function $\alpha \in L^1_+([0, b])$ such that for every $x \in E$ we have

$$\|F(t, x)\| \leq \alpha(t)(1 + \|x\|) \quad a.e.\ t \in [0, b];$$

$(F4)$ there exists a function $k \in L^1_+([0, b])$ such that for every bounded $D \subset E$

$$\chi(F(t, D)) \leq k(t)\, \chi(D)\ ,$$

for a.e. $t \in [0, b]$, where χ is the Hausdorff MNC.

Remark 1. We note that if multifunction $F(\cdot, x)$ is strongly measurable for every $x \in E$, then condition $(F1)$ is fulfilled.

Moreover, we observe that, under conditions $(F1)$ and $(F3)$, for every strongly measurable function $q : [0, b] \to E$ there exists a strongly measurable selection f : $[0, b] \to E$ of the multifunction F: $[0, b] \to K(E)$, F$(t) = F(t, q(t))$ (cf. [16], Theorem 1.3.5). Therefore, under hypothesis $(F2)$, we have that the set $\mathcal{S}^1_{F(\cdot, q(\cdot))\,,\,[0,b]}$ is nonempty.

For the sake of simplicity, we put $J_0 = [0, t_1]\ ;\ J_k =]t_k, t_{k+1}]\,,\ k = 1, \cdots, m$.
Afterwards, in order to define a mild solution of problem (P), we introduce the set

$$\Lambda = \{x : [0, b] \to E\ :\ x_{|\,J_k} \in C(J_k, E)\,,\ k = 0, \cdots, m \text{ and there exists } x(t_k^+) \in E\,,\ k = 1, \cdots, m\}\,.$$

It is easy to verify that $(\Lambda, \|\cdot\|_\Lambda)$ is a Banach space, where, for every $x \in \Lambda$, we put

$$\|x\|_\Lambda = \max\{\|x_k\|_\infty\,,\ k = 0 \cdots, m\}$$

being $x_0 = x_{| J_0}$ and, for any $k = 1, \cdots, m$,

$$x_k(t) = \begin{cases} x(t) & , \ t \in J_k \\ \\ x(t_k^+) & , \ t = t_k \ . \end{cases}$$

Definition 1. A function $x \in \Lambda$ absolutely continuous in the closed interval J_0 and in any interval J_k, $k = 1, \cdots, m$, is a *mild solution* of (P) if

(i) $x(t) = T(t,0)a + \displaystyle\int_0^t T(t,s)f(s)ds + \sum_{0<t_k<t} T(t,t_k)I_k(x(t_k)) \,, \ t \in [0,b] \,, \, k = 1, \cdots, m$

where $f \in \mathcal{S}^1_{F(\cdot,x(\cdot))\,,\,[0,b]}$

(ii) $x(0) = a$

(iii) $x(t_k^+) = x(t_k) + I_k(x(t_k)) \,, \ k = 1, \cdots, m.$

In order to prove our result, we recall a global existence theorem obtained in [10] for the *non impulsive* Cauchy problem

$$(\tilde{P}) \begin{cases} x'(t) \in A(t)x(t) + F(t,x(t)) & , \quad \text{a.e. } t \in [0,d] \\ \\ x(0) = a \in E \end{cases}$$

where d is a fixed positive number.

We also recall that a function $x \in C([0,h];E)$, $0 < h \leq d$, is a mild solution for $(\tilde{P})$ if

(j) $x(t) = T(t,0)a + \displaystyle\int_0^t T(t,s)f(s)ds, \quad t \in [0,h]$

where $f \in \mathcal{S}^1_{F(\cdot,x(\cdot))\,,\,[0,h]}$

(jj) $x(0) = a$.

Theorem 1. ([10], Theorem 4) *Suppose that hypotheses* (A) *and* (F) *are satisfied. Then the set of all (global) mild solutions for non impulsive problem* $(\tilde{P})$ *is a nonempty and compact subset of the space* $C([0,d];E)$.

Now we state and prove our main result.

Theorem 2. *Under assumptions* (A) *and* (F) *there exists a mild solution for the impulsive Cauchy problem* (P).

Proof. We proceed by steps.

Step 1. Let us consider the real interval J_0 and the related Cauchy problem

$$(P_0)\begin{cases} x'(t) \in A(t)x(t) + F(t, x(t)) \quad , \qquad \text{a.e. } t \in J_0 \\ \\ x(0) = a \in E \end{cases}$$

By applying Theorem 1, we can say that there exists a mild solution for non impulsive problem (P_0) on the whole interval J_0.

Step 2. Let $x^0 : J_0 \to E$ be a (global) mild solution for non impulsive problem (P_0).

Then, we have

$$x^0(t) = T(t,0)a + \int_0^t T(t,s)f^0(s)\,ds \ , \quad t \in J_0 \tag{1}$$

where $f^0 \in \mathcal{S}^1_{F(\cdot,x^0(\cdot))\,,\,J_0}$, and hence

$$x^0(t_1) = T(t_1,0)a + \int_0^{t_1} T(t_1,s)f^0(s)\,ds \ .$$

Now, let us consider the interval $\overline{J_1}$ and the Cauchy problem

$$(P_1)\begin{cases} x'(t) \in A(t)x(t) + F(t, x(t)) \quad , \qquad \text{a.e. } t \in \overline{J_1} \\ \\ x(t_1) = x^0(t_1) + I_1(x^0(t_1)) \quad . \end{cases}$$

To prove the existence of a (global) mild solution for non impulsive problem (P_1) we introduce the integral multioperator $\Gamma_1 : C(\overline{J_1}; E) \to P(C(\overline{J_1}; E))$ defined as

$$\Gamma_1(x) = \Big\{ y \in C(\overline{J_1}; E) \quad : \quad y(t) = T(t,0)a + \int_0^{t_1} T(t,s) f^0(s)\, ds + \tag{2}$$

$$+ \int_{t_1}^{t} T(t,s) f(s)\, ds + T(t,t_1) I_1(x^0(t_1))\,,\ f \in \mathcal{S}^1_{F(\cdot,x(\cdot))\,,\,\overline{J_1}} \Big\} \ .$$

Of course, the set of all mild solutions for non impulsive problem (P_1) on $\overline{J_1}$ is the set $Fix\,\Gamma_1 = \{x : x \in \Gamma_1(x)\}$. In order to provide that this set is nonempty, we will show, by following the lines of the proves of Theorems 3 and 4 of [10], that the integral multioperator Γ_1 satisfies all the hypotheses of Corollary 3.3.1 of [16].

First of all, it is easy to check that Γ_1 has convex values.

Then, Γ_1 has closed graph in $C(\overline{J_1}; E) \times C(\overline{J_1}; E)$. In fact, let $(x_n)_{n=1}^{+\infty}$, $(z_n)_{n=1}^{+\infty}$ be sequences in $C(\overline{J_1}; E)$ such that $x_n \to x^*$, $z_n \in \Gamma_1(x_n)$ for $n \geq 1$, $z_n \to z^*$. Moreover, let $(f_n)_{n=1}^{+\infty}$, $f_n \in L^1(\overline{J_1}; E)$, be an arbitrary sequence such that $f_n \in \mathcal{S}^1_{F(\cdot,x_n(\cdot))\,,\,\overline{J_1}}$, for $n \geq 1$. Being the set $\{f_n\}_{n=1}^{+\infty}$ integrably bounded and, for a.e. $t \in \overline{J_1}$, the set $\{f_n(t)\}_{n=1}^{+\infty}$ relatively compact, we can say that $\{f_n\}_{n=1}^{+\infty}$ is semicompact (see the proof of Theorem 3 of [10]). Taking into account Proposition 4.2.1 of [16], we have that set $\{f_n\}_{n=1}^{+\infty}$ is weakly compact in $L^1(\overline{J_1}; E)$, so w.l.o.g. we can assume $f_n \rightharpoonup f^*$ in $L^1(\overline{J_1}; E)$.

Now, let us consider the generalized Cauchy operator on the interval $J_0 \cup J_1$, i.e. the operator

$G : L^1(J_0 \cup J_1; E) \to C(J_0 \cup J_1; E)$ defined by

$$Gf(t) = \int_0^t T(t,s)f(s)\,ds \quad , \; t \in J_0 \cup J_1 \; . \tag{3}$$

From Lemma 1, we know that G satisfies properties $(G1)$ and $(G2)$ on the whole interval $J_0 \cup J_1$.

Afterwards, for any $n \geq 1$, let us define function $\tilde{f}_n : J_0 \cup J_1 \to E$ as

$$\tilde{f}_n(t) = \begin{cases} f^0(t)\,, & t \in [0, t_1[\\ \\ f_n(t)\,, & t \in \overline{J_1}\;, \end{cases} \tag{4}$$

and function $\tilde{x}_n : J_0 \cup J_1 \to E$ as

$$\tilde{x}_n(t) = \begin{cases} x^0(t)\,, & t \in [0, t_1[\\ \\ x_n(t)\,, & t \in \overline{J_1}\;, \end{cases} \tag{5}$$

where f^0 and x^0 are respectively the summable function and the mild solution of problem (P_0) fixed in (1).

Of course, $\tilde{f}_n \in \mathcal{S}^1_{F(\cdot,\tilde{x}_n(\cdot))\,,\,J_0\cup J_1}$.

Finally, let $\tilde{f}^* : J_0 \cup J_1 \to E$ be the summable function defined by

$$\tilde{f}^*(t) = \begin{cases} f^0(t)\,, & t \in [0, t_1[\\ \\ f^*(t)\,, & t \in \overline{J_1}\;. \end{cases}$$

Note that set $\{\tilde{f}_n\}_{n=1}^{+\infty}$ is also semicompact and sequence $(\tilde{f}_n)_{n=1}^{+\infty}$ weakly converges to $\tilde{f}^*$ in $L^1(J_0 \cup J_1; E)$. Therefore, by applying Lemma 2, for the generalized Cauchy operator G of (3) we have in $C(J_0 \cup J_1; E)$ the convergence

$$G\tilde{f}_n \to G\tilde{f}^* \; . \tag{6}$$

By means of (2), (4) and (3), for all $t \in \overline{J_1}$ we can write

$$\begin{aligned} z_n(t) &= T(t,0)a + \int_0^{t_1} T(t,s) f^0(s)\, ds + \int_{t_1}^{t} T(t,s) f_n(s)\, ds + T(t,t_1) I_1(x^0(t_1)) = \\ &= T(t,0)a + \int_0^{t} T(t,s) \tilde{f}_n(s)\, ds + T(t,t_1) I_1(x^0(t_1)) = \\ &= T(t,0)a + G\tilde{f}_n(t) + T(t,t_1) I_1(x^0(t_1)) \ . \end{aligned} \tag{7}$$

By applying (6), we deduce

$$z_n \to T(\cdot,0)a + G\tilde{f}^* + T(\cdot,t_1) I_1(x^0(t_1))$$

in $C(\overline{J_1};E)$ and, from the uniqueness of the limit algorithm, for all $t \in \overline{J_1}$ we obtain

$$\begin{aligned} z^*(t) &= T(t,0)a + G\tilde{f}^*(t) + T(t,t_1) I_1(x^0(t_1)) = \\ &= T(t,0)a + \int_0^{t_1} T(t,s) f^0(s)\, ds + \int_{t_1}^{t} T(t,s) f^*(s)\, ds + T(t,t_1) I_1(x^0(t_1)) \ . \end{aligned}$$

Then we get $f^* \in \mathcal{S}^1_{F(\cdot,x^*(\cdot)),\overline{J_1}}$ (see [16], Lemma 5.1.1); therefore $z^* \in \Gamma_1(x^*)$. Hence Γ_1 is closed.

With the same technique as above and by following the proof of Theorem 3 in [10], we obtain that Γ_1 has compact values.

Now, we prove that the integral multioperator Γ_1 is condensing on bounded sets with respect to the well defined, monotone, nonsingular, regular MNC ν_1 in the space $C(\overline{J_1};E)$ defined by (see [16], Example 2.1.4, or [10], Theorem 3)

$$\nu_1(\Omega) = \max_{\mathcal{D} \in \delta(\Omega)} (\gamma_1(\mathcal{D}), \mathrm{mod}_{C(\overline{J_1};E)}(\mathcal{D}))$$

where:

$\delta(\Omega)$ is the collection of all the denumerable subsets of $\Omega \subset C(\overline{J_1};E)$;

γ_1 is the real MNC defined as

$$\gamma_1(\mathcal{D}) = \sup_{t\in\overline{J_1}} e^{-Lt}\chi(\mathcal{D}(t))$$

with $\mathcal{D}(t) = \{x(t) \,:\, x \in \mathcal{D}\}\,, t \in \overline{J_1}$;

$\mathrm{mod}_{C(\overline{J_1};E)}(\mathcal{D})$ is the modulus of equicontinuity of the set of functions $\mathcal{D}$ given by the formula

$$\mathrm{mod}_{C(\overline{J_1};E)}(\mathcal{D}) = \lim_{\delta\to 0} \sup_{x\in\mathcal{D}} \max_{|t'-t''|\leq\delta} \|x(t') - x(t'')\| \ ;$$

$L > 0$ is a constant chosen so that

$$q := 2D \sup_{t\in\overline{J_1}} \int_0^t e^{-L(t-s)} k(s)\, ds \ < \ 1 \tag{8}$$

(here $k(\cdot)$ is the summable function of assumption $(F4)$ and $D > 0$ is the constant such that

$$\|T(t,s)\|_{\mathcal{L}(E)} \leq D\,, \quad (t,s) \in \Delta, \tag{9}$$

which exists because of the strongly continuity of the evolution operator T on the compact set Δ).

Let $\Omega \subset C(\overline{J_1};E)$ be a bounded set such that

$$\nu_1(\Gamma_1(\Omega)) \geq \nu_1(\Omega) \ . \tag{10}$$

We have to prove that Ω is relatively compact. To this aim, bearing in mind the regularity of ν_1, it will be enough to prove that $\nu_1(\Omega) = (0,0)$.

Let $\{y_n\}_{n=1}^{+\infty} \subset \Gamma_1(\Omega)$ be the denumerable set which achieves that maximum $\nu_1(\Gamma_1(\Omega))$, i.e.

$$\nu_1(\Gamma_1(\Omega)) = (\gamma_1(\{y_n\}_{n=1}^{+\infty}), \mathrm{mod}_{C(\overline{J_1};E)}(\{y_n\}_{n=1}^{+\infty})).$$

Then there exists a set $\{x_n\}_{n=1}^{+\infty} \subset \Omega$ such that $y_n \in \Gamma_1(x_n)\,,\ n \geq 1$; i.e. by using (2)

$$y_n(t) = T(t,0)a + \int_0^{t_1} T(t,s)f^0(s)\, ds + \int_{t_1}^{t} T(t,s)f_n(s)\, ds + T(t,t_1)I_1(x^0(t_1))$$

where $f_n \in \mathcal{S}^1_{F(\cdot,x_n(\cdot))\,,\,\overline{J_1}}$.

With the same arguments as in the proof of Theorem 3 in [10], we can deduce

$$\gamma_1\left(\{y_n\}_{n=1}^{+\infty}\right) \geq \gamma_1\left(\{x_n\}_{n=1}^{+\infty}\right) \ . \tag{11}$$

We give an estimate for $\gamma_1\left(\{y_n\}_{n=1}^{+\infty}\right)$.

Fixed $t \in \overline{J_1}$, by using condition $(F4)$, for all $s \in [0,t]$ we have

$$\begin{aligned}
\chi\left(\{\tilde{f}_n(s)\}_{n=1}^{+\infty}\right) &\leq \chi\left(F(s,\{\tilde{x}_n(s)\}_{n=1}^{+\infty})\right) \leq k(s)\,\chi\left(\{\tilde{x}_n(s)\}_{n=1}^{+\infty}\right) = \\
&= \begin{cases} k(s)\chi(x^0(s)) = 0\ , & s \in [0,t_1[\\ & \\ k(s)\chi(\{x_n(s)\}_{n=1}^{+\infty})\ , & s \in \overline{J_1} \end{cases} \leq \\
&\leq e^{Ls}\,k(s)\,\sup_{\xi\in\overline{J_1}} e^{-L\xi}\,\chi\left(\{x_n(\xi)\}_{n=1}^{+\infty}\right) = e^{Ls}\,k(s)\,\gamma_1\left(\{x_n\}_{n=1}^{+\infty}\right)\ ,
\end{aligned}$$

where $\tilde{f}_n$ and $\tilde{x}_n$ are defined respectively as in (4) and in (5).

Now, set $\{\tilde{f}_n\}_{n=1}^{\infty}$ is integrably bounded. In fact, by using condition $(F3)$, for every $t \in \overline{J_1}$ we have

$$\begin{aligned}
\|\tilde{f}_n(t)\| &\leq \|F(t,\tilde{x}_n(t))\| \leq \alpha(t)(1+\|\tilde{x}_n(t)\|) = \\
&= \begin{cases} \alpha(t)(1+\|x^0(t)\|)\ , & t \in [0,t_1[\\ & \\ \alpha(t)(1+\|x_n(t)\|)\ , & t \in \overline{J_1}\ . \end{cases}
\end{aligned}$$

The continuity of x^0 in J_0 and the boundedness of set $\{x_n\}_{n=1}^{\infty} \subset \Omega$ leads to the integrably boundedness of $\{\tilde{f}_n\}_{n=1}^{\infty}$.

Since set $\{\tilde{f}_n\}_{n=1}^{+\infty}$ and generalized Cauchy operator G defined by (3) satisfy the hypotheses

of Lemma 3, we can say

$$\chi\left(\{G\tilde{f}_n(t)\}_{n=1}^{+\infty}\right) \leq 2D\int_0^t e^{Ls}\,k(s)\,\gamma_1\left(\{x_n\}_{n=1}^{+\infty}\right)\,ds$$
$$\leq 2D\,\gamma_1\left(\{x_n\}_{n=1}^{+\infty}\right)\int_0^t e^{Ls}\,k(s)\,ds\ . \qquad (12)$$

where D was defined in (9).

By proceeding in the same way as in (7) for z_n, we can write

$$y_n(t) = T(t,0)a + G\tilde{f}_n(t) + T(t,t_1)I_1(x^0(t_1))\,, \quad t\in\overline{J_1}\,,\ n\in I\!N\ , \qquad (13)$$

so that

$$\gamma_1\left(\{y_n\}_{n=1}^{\infty}\right) = \gamma_1\left(\{G\tilde{f}_n\}_{n=1}^{\infty}\right)\ .$$

Hence, by using also (11), (12) and (8), as in Theorem 3 in [10] we have

$$\begin{aligned}\gamma_1\left(\{x_n\}_{n=1}^{+\infty}\right) &\leq \gamma_1\left(\{y_n\}_{n=1}^{+\infty}\right) = \gamma_1\left(\{G\tilde{f}_n\}_{n=1}^{+\infty}\right) = \\ &= \sup_{t\in\overline{J_1}} e^{-Lt}\chi\left(\{G\tilde{f}_n(t)\}_{n=1}^{+\infty}\right) \leq \sup_{t\in\overline{J_1}} e^{-Lt}\,2D\,\gamma_1\left(\{x_n\}_{n=1}^{+\infty}\right)\int_0^t e^{Ls}\,k(s)\,ds = \\ &= 2D\,\gamma_1\left(\{x_n\}_{n=1}^{+\infty}\right)\sup_{t\in\overline{J_1}}\int_0^t e^{-L(t-s)}\,k(s)\,ds = q\,\gamma_1\left(\{x_n\}_{n=1}^{+\infty}\right)\ .\end{aligned}$$

By using (8) again, we have

$$\gamma_1\left(\{x_n\}_{n=1}^{+\infty}\right) = 0$$

and then

$$\chi\left(\{x_n(t)\}_{n=1}^{+\infty}\right) = 0 \ \text{ for every } t\in\overline{J_1}\,.$$

By using the last equality and hypotheses $(F3)$ and $(F4)$ we can prove that set $\{\tilde{f}_n\}_{n=1}^{+\infty}$ is semicompact. Now, by applying Lemma 1 and Lemma 2, we can conclude that set $\{G\tilde{f}_n\}_{n=1}^{+\infty}$

is relatively compact in $C(J_0 \cup J_1; E)$. The representation of y_n given by (13) yields that set $\{y_n\}_{n=1}^{+\infty}$ is also relatively compact in $C(\overline{J_1}; E)$.

Besides we have $\nu_1(\Gamma_1(\Omega)) = (0,0)$ and so, by using (10), we can write

$$\nu_1(\Omega) = (0,0) \ .$$

Therefore Ω is relatively compact.

We can suppose that constant L of (8) is chosen so that

$$\max_{t \in \overline{J_1}} D \int_{t_1}^{t} e^{-L(t-s)} \alpha(s)\, ds \ = \ q^* \ < \ 1 \tag{14}$$

(D is from (9) and α is the summable positive function of assumption ($F3$)).

By following the lines of the proof of Theorem 4 of [10], we consider in $C(\overline{J_1}; E)$ the equivalent norm defined as

$$\|x\|_* = \max_{t \in \overline{J_1}} e^{-Lt} \|x(t)\|$$

and we fix a radius

$$r \geq D \left(\|a\| + \|I_1(x^0(t_1))\| + 2\|\alpha\|_{L^1_+([0,b])} + \|\alpha\|_{L^1_+([0,b])} b \|x^0\|_{C(J_0;E)} \right) (1 - q^*)^{-1}$$

or, equivalently,

$$D \left(\|a\| + \|I_1(x^0(t_1))\| + 2\|\alpha\|_{L^1_+([0,b])} + b\|\alpha\|_{L^1_+([0,b])} \|x^0\|_{C(J_0;E)} \right) + r\, q^* \leq r \, . \tag{15}$$

Now, if

$$\bar{B}_r(0) = \{ x \in C(\overline{J_1}; E) \ : \ \|x\|_* \leq r \}$$

is the closed ball in the space $(C(\overline{J_1}; E), \| \cdot \|_*)$, we prove that Γ_1 maps $\bar{B}_r(0)$ into itself.

Let us consider arbitrary $x \in \bar{B}_r(0)$ and $y \in \Gamma_1(x)$. Let $f \in \mathcal{S}^1_{F(\cdot, x(\cdot))\, ,\, \overline{J_1}}$ be a function such that

$$y(t) = T(t,0)a + \int_0^{t_1} T(t,s)\, f^0(s)\, ds + \int_{t_1}^{t} T(t,s)\, f(s)\, ds + T(t,t_1) I_1(x^0(t_1)) \quad , \ t \in \overline{J_1} \ .$$

Then for any $t \in \overline{J_1}$, by means of (9), $(F3)$, (14) and (15), we have

$$\begin{aligned}
e^{-Lt}\|y(t)\| &\le e^{-Lt}\|T(t,0)\|_{\mathcal{L}(E)}\,\|a\| + e^{-Lt}\int_0^{t_1}\|T(t,s)\|_{\mathcal{L}(E)}\,\|f^0(s)\|\,ds\, + \\
&\quad + e^{-Lt}\int_{t_1}^{t}\|T(t,s)\|_{\mathcal{L}(E)}\,\|f(s)\|\,ds + e^{-Lt}\|T(t,t_1)\|_{\mathcal{L}(E)}\,\|I_1(x^0(t_1))\| \le \\
&\le e^{-Lt}D\,\|a\| + e^{-Lt}\int_0^{t_1} D\,\alpha(s)(1+\|x^0(s)\|)\,ds\, + \\
&\quad + e^{-Lt}\int_{t_1}^{t} D\,\alpha(s)(1+\|x(s)\|)\,ds + e^{-Lt}D\,\|I_1(x^0(t_1))\| \le \\
&\le D\,\Big(\|a\| + \|I_1(x^0(t_1))\|\Big) + 2D\|\alpha\|_{L^1_+([0,b])}\, + \\
&\quad + e^{-Lt}D\left(\int_0^{t_1}\alpha(s)\,\|x^0(s)\|\,ds + \int_{t_1}^{t}\alpha(s)\,\|x(s)\|\,ds\right) = \\
&= D\,\Big(\|a\| + \|I_1(x^0(t_1))\| + 2\|\alpha\|_{L^1_+([0,b])}\Big) + \\
&\quad + D\left(e^{-Lt}\int_0^{t_1}\alpha(s)\,\|x^0(s)\|\,ds + \int_{t_1}^{t} e^{-L(t-s)}\alpha(s)e^{-Ls}\,\|x(s)\|\,ds\right) \le \\
&\le D\,\Big(\|a\| + \|I_1(x^0(t_1))\| + 2\|\alpha\|_{L^1_+([0,b])}\Big) + \\
&\quad + D\|\alpha\|_{L^1_+([0,b])}t_1\|x^0\|_{C(J_0;E)} + D\|x\|_*\int_{t_1}^{t} e^{-L(t-s)}\alpha(s)\,ds \le \\
&\le D\,\Big(\|a\| + \|I_1(x^0(t_1))\| + 2\|\alpha\|_{L^1_+([0,b])} + b\,\|\alpha\|_{L^1_+([0,b])}\,\|x^0\|_{C(J_0;E)}\Big) + r\,q^* \le r\,.
\end{aligned}$$

Hence, $\|y\|_* \le r$.

All the assumptions of the fixed point theorem ([16], Corollary 3.3.1) are satisfied, so we can conclude that there exists a (global) mild solution for non impulsive problem (P_1), say

$$x^1(t) = T(t,0)a + \int_0^{t_1} T(t,s)f^0(s)\,ds + \int_{t_1}^{t} T(t,s)f^1(s)\,ds + T(t,t_1)I_1(x^0(t_1))\ ,\ t \in \overline{J_1}$$

where $f^1 \in \mathcal{S}^1_{F(\cdot,x^1(\cdot))\,,\,\overline{J_1}}$.

Step 3. Going on in the same way as in Step 2, in any interval J_k , $k = 2, \cdots, m$, we achieve as many mild solutions

$$x^k(t) = T(t,0)a + \int_0^{t_1} T(t,s)f^0(s)\,ds + \cdots + \int_{t_k}^{t} T(t,s)f^k(s)\,ds\, +$$

$$+T(t,t_1)I_1(x^0(t_1)) + \cdots + T(t,t_k)I_k(x^{k-1}(t_k)) \ , \ t \in \overline{J_k} \ , \ k = 2, \cdots, m \tag{16}$$

where $f^k \in \mathcal{S}^1_{F(\cdot,x^k(\cdot))\,,\,\overline{J_k}}$.

Now, if we consider the function $x : [0,b] \to E$ defined by

$$x(t) = \begin{cases} x^0(t) \ , & t \in J_0 \\ \\ x^k(t) \ , & t \in J_k \, , \, k = 1, \cdots, m \ , \end{cases}$$

it is easy to observe that $x(0) = a$ and $x(t_k^+) = x(t_k) + I_k(x(t_k))\, ,\ k = 1, \cdots, m$. Moreover, put $f : [0,b] \to E$ the function defined as

$$f(t) = \begin{cases} f^0(t) \ , & t \in J_0 \\ \\ f^k(t) \ , & t \in J_k \, , \, k = 1, \cdots, m \ , \end{cases}$$

where function f^0 is from (1) and functions f^k are from (16) (of course, $f \in \mathcal{S}^1_{F(\cdot,x(\cdot))\,,\,[0,b]}$), we have

$$x(t) = T(t,0)a + \int_0^t T(t,s)f(s)ds + \sum_{0<t_k<t} T(t,t_k)I_k(x(t_k))\, , \ t \in [0,b] \, , \, k = 1, \cdots, m \ .$$

Then, we can conclude that x is a mild solution for impulsive Cauchy problem (P). □

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Two-Step Iterative Algorithms and Applications

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Abstract

First a general framework for two-step iterative algorithms is introduced and then it is applied to the approximation solvability of a system of two nonlinear variational inequality (SNVI) problems. Let K be a nonempty closed convex subset of a real Hilbert space H. The $SNVI$ problem is stated as follows: find elements $x^*, y^* \in K$ such that

$$\langle \rho T(y^*) + x^* - y^*, x - x^* \rangle \geq 0 \text{ for all } x \in K,$$

$$\langle \eta T(x^*) + y^* - x^*, x - y^* \rangle \geq 0 \text{ for all } x \in K,$$

where $T : K \to H$ is a nonlinear mapping and ρ and η are positive constants.

Mathematics Subject Classifications: 49J40, 65B05

Key Words and Phrases: System of relaxed cocoercive variational inequalities, projection methods, strongly monotone mappings, approximation solvability

1. Introduction

Recently, one of the authors [13] introduced and studied a two-step model for projection methods in the context of investigating the approximation solvability of a system of two nonlinear variational inequality problems involving strongly monotone mappings. The study of this nature was followed by recent publications [2,6]. The notion of the relaxed cocoercivity - a relatively new concept

- is more general than cocoercivity and strong monotonicity, while the cocoercivity (also known as the Dunn property) itself was initiated in the context of studying gradient processes. The cocoercivity is more general than the strong monotonicity, which is well-explored in a wide range of problems arising from computational/numerical mathematics as well as from other branches of science and engineering. Projection methods and their variants have been applied widely under suitable constraints to problems stemming, especially from complementarity problems, convex quadratic programming, and other variational problems. Here, in this paper, we develop two-step general frameworks for iterative algorithms, which contain a number of known iterative procedures as special versions, and then consider, based on the convergence of these two-step general frameworks for projection methods, the approximation solvability of a system of two nonlinear relaxed cocoercive variational inequality problems in a Hilbert space setting. We also present some examples of interest on basic auxiliary notions. The obtained results extend/generalize results in [2,6,13,20], and others. For more details on general variational inequality problems and projection methods, we refer to [1-25].

Let H be a real Hilbert space with the inner product $\langle x, y\rangle$ and norm $\|x\|$ for all $x, y \in H$. Let $T : K \to H$ be any mapping on K and K be a nonempty closed convex subset of H. We consider a system of two nonlinear variational inequality (abbreviated as SNVI) problems as follows: determine elements $x^*, y^* \in K$ such that

$$\langle \rho T(y^*) + x^* - y^*, x - x^*\rangle \geq 0 \text{ for all } x \in K, \tag{1}$$

$$\langle \eta T(x^*) + y^* - x^*, x - y^*\rangle \geq 0 \text{ for all } x \in K, \tag{2}$$

where $\rho, \eta > 0$.

The $SNVI(1)-(2)$ problem is equivalent to the following system of projection formulas:

$$x^* = P_K[y^* - \rho T(y^*)], \tag{3}$$

$$y^* = P_K[x^* - \eta T(x^*)], \tag{4}$$

where ρ and η are positive constants and P_K is the projection of H onto K.

We note that for $\eta = 0$, the $SNVI(1) - (2)$ problem reduces to the NVI problem: determine an element $x^* \in K$ such that

$$\langle T(x^*), x - x^*\rangle \geq 0 \text{ for all } x \in K. \tag{5}$$

Let K be a closed convex cone of H. The $SNVI(1)-(2)$ problem is equivalent to a system of nonlinear complementarity (abbreviated as SNC) problems: find elements $x^*, y^* \in K$ such that $T(x^*), T(y^*) \in K^*$ and

$$\langle \rho T(y^*) + x^* - y^*, x^*\rangle = 0, \tag{6}$$

$$\langle \eta T(x^*) + y^* - x^*, y^* \rangle = 0, \tag{7}$$

where K^* is a polar cone to K defined by

$$K^* = \{f \in K : \langle f, x \rangle \geq 0 \text{ for all } x \in K\}.$$

Proposition 1. Let K be a closed convex cone of H. Then $SNVI(1) - (2)$ and $SNC(6) - (7)$ problems have the same set of solutions.

Now we recall the following auxiliary results for the approximation solvability of nonlinear variational inequality problems based on iterative procedures.

Lemma 1. For an element $z \in H$, we have

$$x \in K \text{ and } \langle x - z, y - x \rangle \geq 0 \text{ for all } y \in K$$

$$\Longleftrightarrow x = P_K(z).$$

Lemma 2. For $u, v \in H$, we have

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u\|^2 - \|v\|^2}{2}.$$

Definition 1. A mapping $T : H \to H$ is called:
(i) monotone if for each $x, y \in H$ we have

$$\langle T(x) - T(y), x - y \rangle \geq 0.$$

(ii) r-strongly monotone if there exists a positive constant r such that

$$\langle T(x) - T(y), x - y \rangle \geq r\|x - y\|^2 \text{ for all } x, y \in H.$$

(iii) r-expansive if

$$\|T(x) - T(y)\| \geq r\|x - y\|.$$

(iv) expansive if r = 1 in (iii).
(v) s-generalized pseudocontraction [10] if (for $s > 0$)

$$\langle T(x) - T(y), x - y \rangle \leq s\|x - y\|^2 \text{ for all } x, y \in H.$$

Example 1. Consider a mapping $T : R^n \to R^n$ defined by

$$T(x) = cI(x) + v,$$

where $x, v \in R^n$ with v fixed, I is the $n \times n$ identity matrix and $c > 0$. Then T is r-strongly monotone for $0 < r \leq c$.

Example 2. Let a function T be defined by

$$T(x) = x^3 \text{ for } x \in [-1, 1].$$

Then T is not strongly monotone.

Definition 2. A mapping $T : H \to H$ is called s-Lipschitz continuous (or Lipschitzian) if there exists a constant $s \geq 0$ such that

$$\|T(x) - T(y)\| \leq s\|x - y\| \text{ for all } x, y \in H.$$

This clearly implies that

$$\langle T(x) - T(y), x - y\rangle \leq s\|x - y\|^2.$$

Proposition 2. Let T be expansive and s-generalized pseudocontractive [10]. Then T is an s-anticocoercive mapping, that is,

$$\langle T(x) - T(y), x - y\rangle \leq s\|T(x) - T(y)\|^2.$$

Proposition 3. Let T be an expansive mapping. Then $I - T$ is $\frac{1}{2} - anticocoercive$, where I is the identity mapping.

When $s = 1$, the s-Lipschitz continuous mapping T is said to be nonexpansive, that is,

$$\|T(x) - T(y)\| \leq \|x - y\|.$$

Definition 3. A mapping $T : H \to H$ is said to be $\mu - cocoercive$ [1] if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y\rangle \geq \mu\|T(x) - T(y)\|^2,$$

where μ is a positive constant.

Example 3. Let $T : H \to H$ be nonexpansive. Then $I - T$ is $\frac{1}{2} - cocoercive$, where I is the identity mapping on H. For if $x, y \in H$, we have

$$\begin{aligned}
\|(I - T)(x) - (I - T)(y)\|^2 &= \|x - y - (T(x) - T(y))\|^2 \\
&= \|x - y\|^2 - 2\langle x - y, T(x) - T(y)\rangle + \|T(x) - T(y)\|^2 \\
&\leq 2\{\|x - y\|^2 - \langle x - y, T(x) - T(y)\rangle\} \\
&= 2\langle x - y, (I - T)(x) - (I - T)(y)\rangle,
\end{aligned}$$

that is,

$$\langle (I-T)(x)-(I-T)(y), x-y\rangle \geq \frac{1}{2}\|(I-T)(x)-(I-T)(y)\|^2.$$

Example 4. [3] Let $T : H \to 2^H$ be a maximal monotone mapping and let J_r denote the resolvent of T for $r > 0$. Then $T_r : H \to H$ is $r-cocoercive$, where $T_r = \frac{1}{r}(I - J_r)$.

Clearly, every $\mu-cocoercive$ mapping T is $\frac{1}{\mu}-Lipschitz$ continuous. We can easily conclude that the following implications on monotonicity, strong monotonicity and expansiveness hold:

$$\begin{array}{c} Strong\,Monotonicity \Longrightarrow Monotonicity \\ \Downarrow \\ Expansiveness \end{array}$$

Definition 4. A mapping $T : H \to H$ is said to be:
(i) relaxed $\gamma - cocoercive$ if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq -\gamma\|T(x) - T(y)\|^2 \text{ for all } x, y \in H.$$

(ii) relaxed $\gamma - r - cocoercive$ if there exist constants $\gamma, r > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq -\gamma\|T(x) - T(y)\|^2 + r\|x - y\|^2 \text{ for all } x, y \in H.$$

Proposition 4. If a mapping $T : H \to H$ is monotone, then T is relaxed $\gamma - cocoercive$ for $\gamma > 0$. The converse may not be true in general.

Proposition 5. Every $\gamma - cocoercive$ mapping $T : H \to H$ is relaxed $\gamma - cocoercive$, while the converse may not hold in general.

Proposition 6. If a mapping $T : H \to H$ is $r - strongly$ monotone, then T is relaxed $\gamma - r - cocoercive$ for $\gamma, r > 0$.

Proposition 7. If a mapping $T : H \to H$ is $\frac{1}{\alpha} - strongly$ monotone and $\frac{1}{\alpha} - Lipschitz$ continuous, then T is $\alpha - cocoercive$.

Example 5. Consider a mapping $T : H \to H$, which is a generalized $r - pseudocontraction$ [10] for $r > 0$. Then $I - T$ is $(1 - r) - strongly$ monotone

for $r < 1$ and relaxed $\gamma - (1-r) - cocoercive$ for $\gamma > 0$. Clearly, $I - T$ is $(1-r) - strongly$ monotone, and hence, we have

$$\langle (I-T)(x) - (I-T)(y), x-y\rangle + \gamma\|(I-T)(x) - (I-T)(y)\|^2 \geq (1-r)\|x-y\|^2,$$

for all $x, y \in H$.
We note that this class of mappings is more general than the class of strongly monotone mappings. As a result we arrive at:

$$The\, r - Strong\ Monotonicity$$

$$\Downarrow$$

$$The\ Relaxed\ \gamma - r - Cocoercivity$$

2. System of Iterative Algorithms

In this section, we present the convergence analysis for a system of projection methods in the context of the approximation solvability of the $SNVI(1) - (2)$ problem.

Algorithm 1. For arbitrarily chosen initial points $x^0, y^0 \in K$, compute sequences $\{x^k\}$ and $\{y^k\}$ such that

$$\begin{aligned} x^{k+1} &= (b^k - a^k)x^k + (1-(b^k-a^k))P_K[y^k - \rho T(y^k)]], && (8)\\ y^k &= P_K[x^k - \eta T(x^k)], && (9) \end{aligned}$$

where P_K is the projection of H onto K and $\rho, \eta > 0$ are constants with

$$0 \leq a^k \leq b^k \leq 1.$$

Algorithm 2. For arbitrarily chosen initial points $x^0, y^0 \in K$, update iteratively sequences $\{x^k\}$ and $\{y^k\}$ such that for $\rho, \eta > 0$ and for $k \geq 0$, we have

$$\begin{aligned} x^{k+1} &= (b^k - a^k)x^k + [1-(b^k-a^k)]P_K[(1-\rho)y^k + \rho T(y^k)] && (10)\\ &= (b^k - a^k)x^k + [1-(b^k-a^k)]P_K[y^k - \rho(I-T)y^k] && (11)\\ y^k &= P_K[x^k - \eta(I-T)(x^k)], && (12) \end{aligned}$$

where

$$0 \leq a^k \leq b^k \leq 1.$$

Algorithm 3. For arbitrarily chosen initial points $x^0, y^0 \in K$, iteratively update sequences $\{x^k\}$ and $\{y^k\}$ such that (for $\rho, \eta > 0$)

$$\begin{aligned} x^{k+1} &= (1-a^k)x^k + a^k P_K[y^k - \rho T(y^k)] &(13)\\ y^k &= P_K[x^k - \eta T(x^k)]. &(14)\end{aligned}$$

Algorithm 4. For arbitrarily chosen initial points $x^0, y^0 \in K$, compute sequences $\{x^k\}$ and $\{y^k\}$ such that (for $\rho > 0$)

$$\begin{aligned} x^{k+1} &= P_K[a^k x^k + (1-a^k)P_K[y^k - \rho T(y^k)]] &(15)\\ y^k &= P_K[x^k - \rho T(x^k)], &(16)\end{aligned}$$

where $0 \leq a^k \leq 1$.

Algorithm 5. For arbitrarily chosen initial points $x^0, y^0 \in K$, update iteratively sequences $\{x^k\}$ and $\{y^k\}$ such that

$$\begin{aligned} x^{k+1} &= a^k x^k + (1-a^k)P_K[x^k - \rho T(y^k)] &(17)\\ y^k &= P_K[x^k - \rho T(x^k)], &(18)\end{aligned}$$

where $\rho > 0$ and $0 \leq a^k \leq 1$.

Algorithm 6. For arbitrarily chosen initial points $x^0, y^0 \in K$, update iteratively sequences $\{x^k\}$ and (y^k) such that

$$\begin{aligned} x^{k+1} &= P_K[x^k - \rho T(y^k)] &(19)\\ y^k &= P_K[x^k - \rho T(x^k)]. &(20)\end{aligned}$$

Algorithm 7. For an arbitrarily chosen initial point $x^0 \in K$, compute the sequence $\{x^k\}$ such that

$$x^{k+1} = P_K[a^k x^k + (1-a^k)P_K[x^k - \rho T(x^k)]], \tag{21}$$

where $0 \leq a^k \leq 1$.

Algorithm 8. For an arbitrarily chosen initial point $x^0 \in K$, iteratively update the sequence $\{x^k\}$ such that

$$x^{k+1} = P_K[x^k - \rho T(x^k)], \tag{22}$$

where ρ is a positive stepsize.

We now present, based on Algorithm 1, the approximation solvability of the $SNVI(1)-(3)$ problem involving $\gamma - cocoercive$ mappings in a Hilbert space setting.

3. Two-Step Approximation Solvability

Theorem 1. Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $T : K \to H$ be relaxed $\gamma - r - cocoercive$ and $\mu - Lipschitz$ continuous. Suppose that the following assumptions hold:

(i) $x^*, y^* \in K$ form a solution to the $SNVI(1) - (2)$ problem.

(ii) Sequences $\{x^k\}$ and $\{y^k\}$ are generated by Algorithm 1.

(iii) Sequences $\{a^k\}$ and $\{b^k\}$ satisfy $0 \leq a^k \leq b^k \leq 1$ and

$$\sum_{k=0}^{\infty} a^k = \infty \, and \, \sum_{k=0}^{\infty} b^k = \infty.$$

Then sequences $\{x^k\}$ and $\{y^k\}$, respectively, converge to x^* and y^* for

$$\sqrt{1 - 2\rho[r - \gamma\mu^2 - \frac{1}{2}\rho\mu^2]} < 1, \, and$$

$$\sqrt{1 - 2\eta[r - \gamma\mu^2 - \frac{1}{2}\eta\mu^2]} < 1.$$

Proof. Since $x^*, y^* \in K$ form a solution to the SNVI (1)-(2) problem, it follows that

$$\begin{aligned} x^* &= P_K[y^* - \rho T(y^*)], \\ y^* &= P_K[x^* - \eta T(x^*)]. \end{aligned}$$

Applying Algorithm 1, we have

$$\begin{aligned} & \|x^{k+1} - x^*\| \\ = \; & \|(b^k - a^k)x^k + (1 - (b^k - a^k))P_K[y^k - \rho T(y^k)] \\ - \; & (b^k - a^k)x^* - (1 - (b^k - a^k))P_K[y^* - \rho T(y^*)]\| \end{aligned}$$

$$
\begin{aligned}
&\leq (b^k - a^k)\|x^k - x^*\| \\
&+ (1-(b^k-a^k))\|P_K[y^k - \rho T(y^k)] - P_K[y^* - \rho T(y^*)]\| \\
&\leq (b^k - a^k)\|x^k - x^*\| + (1-(b^k-a^k))\|y^k - y^* - \rho[T(y^k) - T(y^*)]\|.
\end{aligned}
$$

Since T is relaxed $\gamma - r - cocoercive$ and $\mu - Lipschitz$ continuous, we have

$$
\begin{aligned}
& \|y^k - y^* - \rho[T(y^k) - T(y^*)]\|^2 \\
= \ & \|y^k - y^*\|^2 - 2\rho\langle T(y^k) - T(y^*), y^k - y^*\rangle \\
+ \ & \rho^2\|T(y^k) - T(y^*)\|^2 \\
\leq \ & \|y^k - y^*\|^2 + 2\rho\gamma\|T(y^k) - T(y^*)\|^2 + \rho^2\|T(y^k) - T(y^*)\|^2 \\
- \ & 2\rho r\|y^k - y^*\|^2 \\
\leq \ & \|y^k - y^*\|^2 - 2\rho r\|y^k - y^*\|^2 \\
+ \ & [2\rho\gamma\mu^2 + \rho^2\mu^2]\|y^k - y^*\|^2 \\
= \ & [1 - 2\rho r + 2\rho\gamma\mu^2 + \rho^2\mu^2]\|y^k - y^*\|^2 \\
= \ & L^2\|y^k - y^*\|^2,
\end{aligned}
$$

where $L = \sqrt{1 - 2\rho(r - \gamma\mu^2 - \frac{1}{2}\rho\mu^2)} < 1$.

It follows from above arguments that

$$
\|x^{k+1} - x^*\| \leq (b^k - a^k)\|x^k - x^*\| + (1-(b^k-a^k))L\|y^k - y^*\|, \tag{23}
$$

for $0 < L < 1$.

Similarly, we have

$$
\|y^k - y^*\|^2 \ = \ \|P_K[x^k - \eta T(x^k)] - P_K[x^* - \eta T(x^*)]\|^2
$$

$$
\begin{aligned}
&\leq \|x^k - x^* - \eta[T(x^k) - T(x^*)]\|^2 \\
&\leq \|x^k - x^*\|^2 + (2\eta\gamma\mu^2 + \eta^2\mu^2)\|x^k - x^*\|^2 \\
&- 2\eta r\|x^k - x^*\|^2 \\
&= M^2\|x^k - x^*\|^2 \\
&\leq \|x^k - x^*\|^2,
\end{aligned}
$$

where $M = \sqrt{1 - 2\eta[r - \gamma\mu^2 - \frac{1}{2}\eta\mu^2]} < 1$.
Therefore, we have

$$\|x^{k+1} - x^*\|$$

$$
\begin{aligned}
&\leq (b^k - a^k)\|x^k - x^*\| + [1 - (b^k - a^k)]L\|x^k - x^*\| \\
&\leq [1 - (1 - L)a^k]\|x^k - x^*\| - (1 - L)(1 - b^k)\|x^k - x^*\| \\
&\leq \prod_{j=1}^{k}[1 - (1 - L)a^j]\|x^1 - x^*\| \\
&+ (1 - L)\prod_{j=1}^{k}(1 - b^j)\|x^1 - x^*\|,
\end{aligned}
$$

where $0 < L < 1$. Since under the hypotheses of the theorem, $\sum_{j=1}^{\infty} a^k$ and $\sum_{j=1}^{\infty} b^k$ diverge, it implies, in light of [22] that

$$\lim_{k\to\infty}\prod_{j=1}^{k}[1 - (1 - L)a^j] = 0,$$

$$\lim_{k\to\infty}\prod_{j=1}^{k}(1 - b^j) = 0.$$

Hence, the sequence $\{x^k\}$ converges to x^* and consequently, the sequence $\{y^k\}$converges to y^*. This completes the proof.

Theorem 2. [14] Let H be a real Hilbert space and K its nonempty closed convex subset. Let $T : K \to H$ be relaxed $\gamma - r - cocoercive$ and $\mu - Lipschitz$ continuous. If $x^*, y^* \in K$ form a solution to the SNVI $(1) - (2)$ problem, if sequences $\{x^k\}$ and $\{y^k\}$ are generated by Algorithm 3, and if $0 \leq a^k \leq 1$ with

$$\sum_{k=0}^{\infty} a^k = \infty,$$

then sequences $\{x^k\}$ and $\{y^k\}$, respectively, converge to x^* and y^* for

$$\sqrt{1 - 2\rho[r - \gamma\mu^2 - \frac{1}{2}\rho\mu^2]} < 1, \; and$$
$$\sqrt{1 - 2\eta[r - \gamma\mu^2 - \frac{1}{2}\eta\mu^2]} < 1.$$

Proof. Since $x^*, y^* \in K$ form a solution to the SNVI (1)-(2) problem, it follows that

$$\begin{aligned} x^* &= P_K[y^* - \rho T(y^*)], \\ y^* &= P_K[x^* - \eta T(x^*)]. \end{aligned}$$

Applying Algorithm 3, we have

$$\begin{aligned} &\|x^{k+1} - x^*\| \\ &= \|(1 - a^k)x^k + a^k P_K[y^k - \rho T(y^k)] \\ &- (1 - a^k)x^* - a^k P_K[y^* - \rho T(y^*)]\| \\ &\leq (1 - a^k)\|x^k - x^*\| \\ &+ a^k \|P_K[y^k - \rho T(y^k)] - P_K[y^* - \rho T(y^*)]\| \\ &\leq (1 - a^k)\|x^k - x^*\| + a^k\|y^k - y^* - \rho[T(y^k) - T(y^*)]\|. \end{aligned}$$

Since T is relaxed $\gamma - r - cocoercive$ and $\mu - Lipschitz$ continuous, we have

$$\|y^k - y^* - \rho[T(y^k) - T(y^*)]\|^2$$

$$
\begin{aligned}
&= \|y^k - y^*\|^2 - 2\rho\langle T(y^k) - T(y^*), y^k - y^*\rangle \\
&+ \rho^2 \|T(y^k) - T(y^*)\|^2 \\
&\le \|y^k - y^*\|^2 + 2\rho\gamma\|T(y^k) - T(y^*)\|^2 + \rho^2\|T(y^k) - T(y^*)\|^2 \\
&- 2\rho r\|y^k - y^*\|^2 \\
&\le \|y^k - y^*\|^2 - 2\rho r\|y^k - y^*\|^2 \\
&+ [2\rho\gamma\mu^2 + \rho^2\mu^2]\|y^k - y^*\|^2 \\
&= [1 - 2\rho r + 2\rho\gamma\mu^2 + \rho^2\mu^2]\|y^k - y^*\|^2 \\
&= L^2\|y^k - y^*\|^2,
\end{aligned}
$$

where $L = \sqrt{1 - 2\rho(r - \gamma\mu^2 - \frac{1}{2}\rho\mu^2)} < 1$. It follows from above arguments that

$$\|x^{k+1} - x^*\| \le (1 - a^k)\|x^k - x^*\| + a^k L\|y^k - y^*\|, \tag{24}$$

for $0 < L < 1$.

Similarly, we have

$$
\begin{aligned}
\|y^k - y^*\|^2 &= \|P_K[x^k - \eta T(x^k)] - P_K[x^* - \eta T(x^*)]\|^2 \\
&\le \|x^k - x^* - \eta[T(x^k) - T(x^*)]\|^2 \\
&\le \|x^k - x^*\|^2 + (2\eta\gamma\mu^2 + \eta^2\mu^2)\|x^k - x^*\|^2 \\
&- 2\eta r\|x^k - x^*\|^2 \\
&= M^2\|x^k - x^*\|^2 \\
&\le \|x^k - x^*\|^2,
\end{aligned}
$$

where $M = \sqrt{1 - 2\eta[r - \gamma\mu^2 - \frac{1}{2}\eta\mu^2]} < 1$.
Therefore, we have

$$
\begin{aligned}
\|x^{k+1} - x^*\| & \\
&\leq (1 - a^k)\|x^k - x^*\| + a^k L\|x^k - x^*\| \\
&\leq [1 - (1 - L)a^k]\|x^k - x^*\| \\
&\leq \prod_{j=1}^{k}[1 - (1 - L)a^j]\|x^1 - x^*\|,
\end{aligned}
$$

where $0 < L < 1$. Since under the hypotheses of the theorem, $\sum_{j=1}^{\infty} a^k$ diverges, it implies, in light of [22] that

$$\lim_{k\to\infty}\prod_{j=1}^{k}[1 - (1 - L)a^j] = 0.$$

Hence, the sequence $\{x^k\}$ converges to x^* and consequently, the sequence $\{y^k\}$converges to y^*. This completes the proof.

Theorem 3. Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \to H$ be $r - Lipschitz$ continuous. Suppose that $x^*, y^* \in K$ form a solution to the SNVI (1)-(2) problem, sequences $\{x^k\}$ and $\{y^k\}$ are generated by Algorithm 2, and $0 \leq a^k \leq b^k \leq 1$ with $\sum_{k=0}^{\infty} a^k$ and $\sum_{k=0}^{\infty} b^k$ divergent. Then sequences $\{x^k\}$ and $\{y^k\}$, respectively, converge to x^* and y^* for

$$\sqrt{1 - 2\rho[(1 - r) - \gamma(1 + r)^2 - \frac{1}{2}\rho(1 + r)^2]} < 1,$$

$$\sqrt{1 - 2\eta[(1 - r) - \gamma(1 + r)^2 - \frac{1}{2}\eta(1 + r)^2]} < 1,$$

for $\gamma > 0$.

Proof. Since T is $r - Lipschitz$ continuous (and hence T is $r - generalized$ pseudocontractive), it implies, in light of [10], that $I - T$ is $(1 - r) - strongly$ monotone for $0 < r < 1$. As a consequence, $I - T$ is relaxed $\gamma - (1 - r) - cocoercive$

for $\gamma > 0$. Now, on applying Algorithm 2, the rest of the proof is similar to that of Theorem 1.

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Approximation formulas for C_0-semigroups and their resolvent operators

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Abstract

We consider an approximation process and the semigroup generated by the differential operator arising from a Voronovskaja-type formula. Besides the representation of the semigroup using the Trotter's approximation formula, here we are interested in describing the resolvent operators in terms of approximation process. These formulas turn out to be useful for some qualitative properties of the semigroup in different meaningful cases.

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1 Resolvent approximation formulas

In different papers it has been considered the possibility of approximating the solution of suitable parabolic problems introducing sequences of positive operators whose iterates evaluated at the initial point converge uniformly to the solution. This possibility is described in detail in [1] and is based on the connection between Voronovskaja type formulas and semigroup theory through Trotter's theorem [10].

However, in order to study different qualitative properties of the semigroup, it seems that the representation in terms of approximating operators of the resolvent operator rather than of the semigroup may play an important role. For this reason, in this paper we give this general description and obtain different formulas of independent interest. As an application we study some qualitative properties of the semigroup generated by some differential operators in different meaningful cases.

We begin with a general result on the approximation of the resolvent operator; if $A : D(A) \to E$ generates a strongly continuous semigroup with growth bound ω, for every $\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda > \omega$ we denote by $R(\lambda, A) : E \to D(A)$ the resolvent operator.

Theorem 1.1 *Let E be a Banach space and $(L_n)_{n\geq 1}$ a sequence of bounded linear operators on E. Assume that there exist $M > 0$, $\omega \in \mathbf{R}$ and a decreasing sequence $(\rho_n)_{n\geq 1}$ of positive real numbers such that $\lim\limits_{n\to+\infty} \rho_n = 0$ and*

$$\|L_n^k\| \leq M\, e^{k\omega\rho_n}\ , \qquad n, k \geq 1\ .$$

Consider the linear operator $A : D(A) \to E$ defined by

$$Au := \lim_{n\to+\infty} \frac{L_n u - u}{\rho_n}\ , \qquad u \in D(A)\ ,$$

where

$$D(A) := \left\{ u \in E \mid \lim_{n\to+\infty} \frac{L_n u - u}{\rho_n} \in E \right\}\ .$$

If $D(A)$ is dense in E and if the range $(\lambda - A)(D(A))$ is dense in E for some $\lambda > \omega$, then A is closable and its closure generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on E which admits the representation $T(t) = \lim\limits_{n\to+\infty} L_n^{[nt]}$ strongly on E.

Moreover, for every $\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda > \omega$ and for every $f \in E$, we have

$$R(\lambda, A)f = \frac{1}{\lambda} \lim_{n\to\infty} \left(f + \sum_{k=0}^{\infty} e^{-\lambda(k+1)\rho_n} (L_n^{k+1} f - L_n^k f) \right)\ . \tag{1.1}$$

Proof. The existence of the semigroup and its representation in terms of iterates of the operators L_n follows fron the Trotter's approximation theorem [10, Theorem 5.1] and therefore we have only to show the last part of the theorem.

Since
$$\left\| e^{-\lambda t} L_n^{[t/\rho_n]} \right\| \le M e^{-\mathrm{Re}\,\lambda t}\, e^{\omega t} = M\, e^{-(\mathrm{Re}\,\lambda - \omega)t} ,$$
the sequence $(e^{-\lambda t} L_n^{[t/\rho_n]})_{n\ge 1}$ is equibounded and is dominated by the summable function $M\, e^{-(\mathrm{Re}\,\lambda-\omega)t}$ over $[0,+\infty[$; hence, from the integral representation of the resolvent (see e.g. [5, Theorem 1.10, (1.13), p. 55] and the dominated convergence theorem we have, for every $f \in E$,

$$\begin{aligned} R(\lambda, A)f &= \int_0^{+\infty} e^{-\lambda t} T(t) f\, dt = \int_0^{+\infty} \lim_{n\to\infty} e^{-\lambda t} L_n^{[t/\rho_n]} f\, dt \\ &= \lim_{n\to\infty} \int_0^{+\infty} e^{-\lambda t} L_n^{[t/\rho_n]} f\, dt \end{aligned} \tag{1.2}$$

and consequently, since $L_n^{[t/\rho_n]}$ is constant on each interval $[k\rho_n, (k+1)\rho_n[$,

$$\begin{aligned} R(\lambda, A)f &= \lim_{n\to\infty} \sum_{k=0}^{\infty} L_n^k f \int_{k\rho_n}^{(k+1)\rho_n} e^{-\lambda t} dt \\ &= \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{e^{-\lambda k\rho_n} - e^{-\lambda(k+1)\rho_n}}{\lambda}\, L_n^k f \\ &= \frac{1}{\lambda} \lim_{n\to\infty} \left(f + \sum_{k=0}^{\infty} e^{-\lambda(k+1)\rho_n} (L_n^{k+1} f - L_n^k f) \right) , \end{aligned}$$

which completes the proof. ∎

We also point out the intermediate representation of the resolvent operator obtained in the preceding proof

$$R(\lambda, A)f = \frac{1}{\lambda} \lim_{n\to\infty} \sum_{k=0}^{\infty} \left(e^{-\lambda k\rho_n} - e^{-\lambda(k+1)\rho_n} \right) L_n^k f . \tag{1.3}$$

Moreover, if for every $n \ge 1$, we have $e^{\lambda/n} \in \rho(L_n)$, then we also have

$$R(\lambda, A)f = \frac{1}{\lambda} \lim_{n\to\infty} (1 - e^{-\lambda\rho_n})\, (I - e^{-\lambda\rho_n} L_n)^{-1} f . \tag{1.4}$$

Indeed, (1.4) is a direct consequence of (1.3) since

$$\begin{aligned} R(\lambda, A)f &= \frac{1}{\lambda} \lim_{n\to\infty} \sum_{k=0}^{\infty} \left(e^{-\lambda k\rho_n} - e^{-\lambda(k+1)\rho_n} \right) L_n^k f \\ &= \frac{1}{\lambda} \lim_{n\to\infty} (1 - e^{-\lambda\rho_n}) \sum_{k=0}^{\infty} e^{-\lambda k\rho_n} L_n^k f \\ &= \frac{1}{\lambda} \lim_{n\to\infty} (1 - e^{-\lambda\rho_n})\, (I - e^{-\lambda\rho_n} L_n)^{-1} f . \end{aligned}$$

In many examples the sequence $(L_n)_{n\geq 1}$ is a bounded approximation process on a compact subset K of $\mathbf{R}^d$ satisfying a Voronovskaja-type formula having the form

$$\lim_{n\to+\infty} n(L_n u - u) = Au$$

for every $u \in C^2(K)$ and hence $\rho_n = 1/n$ and $\omega = 0$; the resolvent representation formula (1.1) in this case becomes

$$R(\lambda, A)f = \frac{1}{\lambda} \lim_{n\to\infty} \left(f + \sum_{k=0}^{\infty} e^{-\lambda(k+1)/n} (L_n^{k+1} f - L_n^k f) \right) \tag{1.5}$$

for every $\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda > 0$; moreover, (1.3) and (1.4) become respectively

$$R(\lambda, A)f = \frac{1}{\lambda} \lim_{n\to\infty} \sum_{k=0}^{\infty} \left(e^{-\lambda k/n} - e^{-\lambda(k+1)/n} \right) L_n^k f \tag{1.6}$$

and, under the further assumption $e^{\lambda/n} \in \rho(L_n)$ for every $n \geq 1$,

$$R(\lambda, A)f = \frac{1}{\lambda} \lim_{n\to\infty} (1 - e^{-\lambda/n})\, (I - e^{-\lambda/n} L_n)^{-1} f\ . \tag{1.7}$$

In the next section we shall consider further representations of the resolvent operator corresponding to particular approximation processes $(L_n)_{n\geq 1}$.

2 Application to qualitative properties of semigroups

2.1 Bernstein operators

Consider the space $C[0,1]$ endowed with the uniform norm and, for every $n \geq 1$, the Bernstein operator $B_n : C[0,1] \to C[0,1]$ defined by setting, for every $f \in C[0,1]$ and $x \in [0,1]$,

$$B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k\, (1-x)^{n-k}\, f\left(\frac{k}{n}\right)\ . \tag{2.1}$$

We recall that B_n can be represented in the diagonal form (see e.g. [3])

$$B_n f = \sum_{j=0}^{n} \lambda_j^{(n)} p_j^{(n)} \mu_j^{(n)}(f), \qquad f \in C[0,1]\ , \tag{2.2}$$

where $\lambda_j^{(n)}$ and $p_j^{(n)}$ are the eigenvalues and respectively the eigenfunctions of B_n and $\mu_j^{(n)}$ are the dual functionals to $p_j^{(n)}$. We also recall that the eigenfunctions $p_j^{(n)}$ are polynomials of degree j and the eigenvalues are given by

$$\lambda_j^{(n)} = \frac{n!}{(n-j)!}\,\frac{1}{n^j}\ , \qquad j = 0, \ldots, n\ ,$$

and satisfy $1 = \lambda_0^{(n)} = \lambda_1^{(n)} > \lambda_2^{(n)} > \ldots > \lambda_n^{(n)}$.

From (2.2) it is easy to obtain, by induction on $k \geq 1$ and using the equality $B_n(p_j^{(n)}) = \lambda_j^{(n)} p_j^{(n)}$,

$$B_n^k f = \sum_{j=0}^{n} (\lambda_j^{(n)})^k p_j^{(n)} \mu_j^{(n)}(f), \qquad f \in C[0,1]\,, \quad k \geq 1\,. \tag{2.3}$$

Moreover, since B_n is a positive contraction we also have $\|B_n^k\| \leq 1$ for every $k \geq 1$.

The differential operator associated with the Voronovskaja formula for the Bernstein operators is given by

$$A_0 u(x) := \frac{x(1-x)}{2}\, u''(x)$$

and generates a C_0-semigroup of positive contractions on the domain

$$D(A_0) := \left\{ u \in C[0,1] \cap C]0,1[\;\mid\; \lim_{x \to 0,1} \frac{x(1-x)}{2}\, u''(x) = 0 \right\}\;;$$

see [1, Chapter 6] for more details.

In [7] Metafune proved that the semigroup generated by $(A_0, D(A_0))$ is analytic.

Here, using Theorem 1.1, we give some explicit formulas of the resolvent operator in terms of Bernstein operators.

Let $\lambda \in \mathbf{C}$ be such that $\operatorname{Re} \lambda > 0$ and consider the resolvent operator $R(\lambda, A_0)$ defined on the space $C([0,1], \mathbf{C})$ of all continuous complex-valued functions on the interval $[0,1]$.

We consider the expression of $R(\lambda, A_0) f$ in the particular case where f is a real-valued polynomial of degree m on the interval $[0,1]$. From (1.2) and (2.3) we can write

$$\begin{aligned}
R(\lambda, A_0) f &= \lim_{n \to +\infty} \int_0^{+\infty} e^{-\lambda t} B_n^{[nt]}(f)\, dt \\
&= \lim_{n \to +\infty} \int_0^{+\infty} e^{-\lambda t} \sum_{j=0}^{n} (\lambda_j^{(n)})^{[nt]} p_j^{(n)} \mu_j^{(n)}(f)\, dt \\
&= \lim_{n \to +\infty} \sum_{j=0}^{n} p_j^{(n)} \mu_j^{(n)}(f) \int_0^{+\infty} e^{-\lambda t} (\lambda_j^{(n)})^{[nt]}\, dt \\
&= \lim_{n \to +\infty} \sum_{j=0}^{m} p_j^{(n)} \mu_j^{(n)}(f) \int_0^{+\infty} e^{-\lambda t} (\lambda_j^{(n)})^{[nt]}\, dt\,.
\end{aligned}$$

From [3, (5.11)] we get, for every $j = 2, \ldots, m$,

$$\lim_{n \to +\infty} (\lambda_j^{(n)})^{[nt]} = e^{-j(j-1)\,t/2} \tag{2.4}$$

and hence, taking into account that $\lambda_0^{[nt]} = \lambda_1^{[nt]} = 1$, by the dominated convergence theorem we can consider the limits $p_j^* := \lim_{n\to+\infty} p_j^{(n)}$ and $\mu_j^* := \lim_{n\to+\infty} \mu_j^{(n)}$ for every $j = 0, \dots, m$ (see [3]) and obtain

$$\begin{aligned} R(\lambda, A_0)f &= \frac{1}{\lambda}(p_0^*\mu_0^*(f) + p_1^*\mu_1^*(f)) + \sum_{j=2}^{m} p_j^*\mu_j^*(f) \int_0^{+\infty} e^{-\lambda t} e^{-j(j-1)\,t/2}\, dt \\ &= \frac{1}{\lambda}\left(p_0^*\mu_0^*(f) + p_1^*\mu_1^*(f) + \sum_{j=2}^{m} \frac{\lambda}{\lambda + \frac{j(j-1)}{2}} p_j^*\mu_j^*(f) \right) . \end{aligned}$$

Then, by [3, Theorem 4.5, Lemma 4.10],

$$\begin{aligned} R(\lambda, A_0)f(x) &= \frac{1}{\lambda}\left(\frac{f(0)+f(1)}{2} + (f(1)-f(0))(x-\frac{1}{2}) \right) \\ &\quad + \sum_{j=2}^{m} \frac{1}{\lambda + \frac{j(j-1)}{2}} x(x-1) \frac{j!(j-2)!}{(2j-2)!} P_{j-2}^{(1,1)}(2x-1) \frac{1}{2}\binom{2j}{j} \times \\ &\quad \times \left\{ (-1)^j f(0) + f(1) - j\int_0^1 f(y) P_{j-2}^{(1,1)}(2y-1)dy \right\} \\ &= \frac{1}{\lambda}\left(\frac{f(0)+f(1)}{2} + (f(1)-f(0))\left(x-\frac{1}{2}\right) \right) \\ &\quad - \sum_{j=2}^{m} \frac{1}{\frac{j(j-1)}{2} + \lambda} x(1-x) \frac{2j-1}{j-1} P_{j-2}^{(1,1)}(2x-1) \times \\ &\quad \times \left\{ (-1)^j f(0) + f(1) - j\int_0^1 f(y) P_{j-2}^{(1,1)}(2y-1)dy \right\}, \end{aligned}$$

where $P_j^{(1,1)}$ denotes as usual the standardized Jacobi polynomial defined by the Rodrigues formula

$$P_j^{(1,1)}(x) := \frac{(-1)^j}{j!\,2^j\,(1-x^2)} \frac{d^j}{dx^j}\left[(1-x^2)^{j+1}\right] .$$

Besides the preceding expression of the resolvent operator, we point out the well-known representation of the semigroup $(T(t))_{t\geq 0}$ generated by A_0 (see, e.g., [6]).

Indeed, from (2.3), we have

$$T(t)f = \lim_{n\to+\infty} \sum_{j=0}^{n} (\lambda_j^{(n)})^{[nt]}\, p_j^{(n)}\, \mu_j^{(n)}(f)$$

strongly on $C[0,1]$ and hence, using (2.4), if f is a polynomial of degree less or

equal to m we obtain, for every $x \in [0,1]$,

$$\begin{aligned}
T(t)f(x) &= \lim_{n\to+\infty} \sum_{j=0}^{m} (\lambda_j^{(n)})^{[nt]}\, p_j^{(n)}(x)\, \mu_j^{(n)}(f) \\
&= \sum_{j=0}^{m} e^{-j(j-1)t/2}\, p_j^{(*)}(x)\, \mu_j^{(*)}(f) \\
&= \frac{f(0)+f(1)}{2} + (f(1)-f(0)) \left(x - \frac{1}{2}\right) \\
&\quad - \sum_{j=2}^{m} e^{-j(j-1)t/2}\, \frac{2j-1}{j-1}\, x(1-x)\, P_{j-2}^{(1,1)}(2x-1) \,\times \\
&\quad \times \left((-1)^j f(0) + f(1) - j \int_0^1 f(y)\, P_{j-2}^{(1,1)}(2y-1)\, dy \right) .
\end{aligned}$$

The above formula allows us to prove the following result.

Theorem 2.1 *The semigroup $(T(t))_{t\geq 0}$ generated by the differential operator $(A_0, D(A_0))$ is immediately compact.*

Proof. Let $t>0$, f a polynomial of degree less than m and $x, y \in [0,1]$. Then

$$\begin{aligned}
T(t)f(x) - T(t)f(y) = (f(1)-f(0))\ (x-y) - \sum_{j=2}^{m} e^{-j(j-1)t/2}\, \frac{2j-1}{j-1} \,\times \\
\times \left[x(1-x) P_{j-2}^{(1,1)}(2x-1) - y(1-y) P_{j-2}^{(1,1)}(2y-1) \right] \times \\
\times \left((-1)^j f(0) + f(1) - j \int_0^1 f(s)\, P_{j-2}^{(1,1)}(2s-1)\, ds \right) .
\end{aligned}$$

By well-known properties of Jacobi polynomials, the derivative of the function

$$g(x) = x(1-x) P_{j-2}^{(1,1)}(2x-1) , \qquad x \in [0,1] ,$$

is given by

$$\begin{aligned}
g'(x) &= (1-2x) P_{j-2}^{(1,1)}(2x-1) + 2x(1-x) (P_{j-2}^{(1,1)})'(2x-1) \\
&= (1-2x) P_{j-2}^{(1,1)}(2x-1) + x(1-x)(j+1) P_{j-3}^{(2,2)}(2x-1),
\end{aligned}$$

and hence

$$\begin{aligned}
|g(x)| &\leq |1-2x||P_{j-2}^{(1,1)}(2x-1)| + x(1-x)(j+1)|P_{j-3}^{(2,2)}(2x-1)| \\
&\leq j-1 + x(1-x)(j+1)|P_{j-3}^{(2,2)}(2x-1)| .
\end{aligned}$$

By [8, Lemma 16], there exists a constant $C>0$ such that for every $x \in [-1,1]$

$$|P_n^{(2,2)}(x)| \leq C(1-x+n^{-2})^{-5/4}(1+x+n^{-2})^{-5/4} .$$

Therefore

$$(1-x)(1+x)|P_{j-3}^{(2,2)}(x)| \le C(j-3)\ .$$

Then we obtain the existence of a constant $K > 0$ such that for every $x \in [0,1]$

$$|g'(x)| \le K(j+1)j\ .$$

Moreover, by the inequality in [9], there exists $K_1 > 0$ such that for every $j = 2, \ldots, m$ and $x \in [0,1]$

$$[x(1-x)]^{3/4} \sqrt{\frac{j(2j-1)}{j-1}} \ \left| P_{j-2}^{(1,1)}(2x-1) \right| \le K_1\ .$$

Consequently, by Lagrange's theorem we get, for all $x, y \in [0,1]$,

$$\begin{aligned} |T(t)f(x) - T(t)f(y)| \ &\le \ 2\|f\|\ |x-y| \\ &\quad + \sum_{j=2}^{m} e^{-\frac{j(j-1)t}{2}} \frac{2j-1}{j-1} Kj(j+1)|x-y| \times \\ &\quad \times \left(2\|f\| + K_1 j \sqrt{\frac{j-1}{j(2j-1)}} \times \right. \\ &\quad \left. \times \|f\| \int_0^1 (s(1-s))^{-\frac{3}{4}} ds \right) \\ &\le \ C\|f\|\ |x-y|, \end{aligned}$$

where C depends only on t. By a density argument, this inequality extends on the whole $C[0,1]$.

At this point, it follows easily that $T(t)$ is a compact operator for all $t > 0$. Indeed, if $(f_j)_{j\ge1}$ is a bounded sequence in $C[0,1]$, then $(T(t)f_j)_{j\ge1}$ is clearly bounded and by the previous considerations, equicontinuous. Hence, a straightforward application of Ascoli-Arzelá's theorem gives the assertion. ∎

Now, we can establish similar results for semigroups generated by suitable perturbations of the differential operator A_0, introducing some simple modifications of the Bernstein operators.

Let $b \in C[0,1]$ be a strictly positive function and for every $n \ge 1$ consider the n-th operator $L_n\colon C[0,1] \to C[0,1]$ given by

$$L_n f := \frac{B_n(f \cdot b)}{b} \tag{2.5}$$

for every $f \in C[0,1]$. Setting $b_{n,k} := b(k/n)$ for all $n \ge 1$ and $k = 0, \ldots, n$, the operator L_n may be explicitly written as

$$L_n f(x) = \frac{1}{b(x)} \sum_{k=0}^{n} \binom{n}{k} b_{n,k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \tag{2.6}$$

for every $f \in C[0,1]$ and $x \in [0,1]$. The operator L_n is clearly positive and by (2.2) it can be represented in diagonal form

$$L_n f = \sum_{k=0}^{n} \lambda_k^{(n)} q_k^{(n)} \nu_k^{(n)}(f) \tag{2.7}$$

for all $f \in C[0,1]$, where $\lambda_k^{(n)}$ and $q_k^{(n)}$ are its eigenvalues and eigenfunctions and $\nu_k^{(n)}$ the dual functionals to $q_k^{(n)}$; from the definition of L_n, we easily get

$$q_k^{(n)} := \frac{p_k^{(n)}}{b} , \qquad \nu_k^{(n)}(f) := \mu_k^{(n)}(f \cdot b) . \tag{2.8}$$

Since the operator L_n is positive, $\|L_n\| = \|L_n(\mathbf{1})\| = \|B_n(b)/b\| \le \|b\|/(\min b)$ and hence the sequence of operators $(L_n)_{n\ge 1}$ is uniformly bounded.

Moreover, for all $f \in C[0,1]$, the sequence $(L_n(f))_{n\ge1}$ converges uniformly to f and if we assume that $b \in C^2[0,1]$, the following Voronovskaja formula holds uniformly in $[0,1]$ for every $f \in C^2[0,1]$

$$\begin{aligned}\lim_{n\to+\infty} n\,(L_n f - f)(x) \quad &= \quad \frac{x(1-x)}{2}\, f''(x) + x(1-x)\,\frac{b'(x)}{b(x)}\, f'(x) \\ &\quad + \frac{x(1-x)}{2}\,\frac{b''(x)}{b(x)}\, f(x) .\end{aligned}$$

Now, consider the differential operator

$$Af(x) := \frac{x(1-x)}{2}\, f''(x) + x(1-x)\,\frac{b'(x)}{b(x)}\, f'(x)$$

with $f \in D(A) := \{u \in C[0,1] \cap C^2]0,1[\mid \lim_{x\to 0^+\,1^-} Au(x) = 0\}$. (We have not included the bounded perturbation $x(1-x)\, b''(x)\, f(x)/(2\, b(x))$ since it does not affect qualitative properties of the semigroup.)

Letting

$$\alpha(x) := \frac{x(1-x)}{2} , \quad \beta(x) := x(1-x)\,\frac{b'(x)}{b(x)} , \qquad x \in [0,1] ,$$

we can consider the functions

$$W(x) := \exp\left(-\int_{1/2}^{x} \frac{\beta(t)}{\alpha(t)}\, dt\right) = \frac{b(1/2)^2}{b(x)^2}$$

and

$$Q(x) := \frac{1}{\alpha(x)W(x)} \int_{1/2}^{x} W(s)\, ds, \qquad R(x) := W(x) \int_{1/2}^{x} \frac{1}{\alpha(s)W(s)}\, ds$$

defined for $x \in]0,1[$. It is easy to check that the function R is integrable over a neighborhood of 0 and 1 while Q is not integrable; thus, according to the Feller

classification (see e.g. [5, pp. 383–404]), the endpoints 0 and 1 are both exit boundary points. Hence $(A, D(A))$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $C[0,1]$ and according to the Trotter's theorem [10, Theorem 5.1], we also have the following representation

$$T(t)f = \lim_{n\to\infty} L_n^{[nt]} f$$

for all $f \in C[0,1]$.

Now, we give an expression of the resolvent operator $R(\lambda, A)f$, with $\lambda \in \mathbf{C}$ be such that $\operatorname{Re}\lambda > 0$ and f a real-valued polynomial of degree m. We have

$$R(\lambda, A)f = \lim_{n\to+\infty} \sum_{j=0}^{m} q_j^{(n)} \nu_j^{(n)}(f) \int_0^{+\infty} e^{-\lambda t} (\lambda_j^{(n)})^{[nt]}\, dt$$

and from [3, (5.11)] we arrive at the formula

$$R(\lambda, A)f = \frac{1}{\lambda}\left(q_0^{(n)}\nu_0^{(n)}(f) + q_1^{(n)}\nu_1^{(n)}(f) + \sum_{j=2}^{m} \frac{\lambda}{\lambda + \frac{j(j-1)}{2}}\, q_j^{(n)}\nu_j^{(n)}(f) \right) .$$

Hence, by [3, Theorem 4.5, Lemma 4.10] and (2.8),

$$\begin{aligned} R(\lambda, A)f(x) \quad &= \quad \frac{1}{\lambda}\left(\frac{f(0)b(0) + f(1)b(1)}{2\,b(x)} + \frac{f(1)b(1) - f(0)b(0)}{b(x)} \left(x - \frac{1}{2} \right) \right) \\ &\quad + \frac{1}{b(x)} \sum_{j=2}^{m} \frac{1}{\frac{j(j-1)}{2} + \lambda}\, x(x-1) \frac{2j-1}{j-1} P_{j-2}^{(1,1)}(2x-1) \;\times \\ &\quad \times \Big((-1)^j f(0)b(0) + f(1)b(1) \\ &\quad - j \int_0^1 f(y)b(y)\, P_{j-2}^{(1,1)}(2y-1) dy \Big) . \end{aligned}$$

Proceeding as above, we are now in a position to state the following result.

Theorem 2.2 *The semigroup generated by the differential operator $(A, D(A))$ is immediately compact.*

Proof. Since the semigroup generated by A is given by

$$\begin{aligned} T(t)f(x) \quad &= \quad \frac{f(0)b(0) + f(1)b(1)}{2b(x)} + \frac{f(1)b(1) - f(0)b(0)}{b(x)} \left(x - \frac{1}{2} \right) \\ &\quad - \frac{1}{b(x)} \sum_{j=2}^{m} e^{-j(j-1)t/2}\, \frac{2j-1}{j-1}\, x(1-x)\, P_{j-2}^{(1,1)}(2x-1) \times \\ &\quad \times \Big((-1)^j f(0)b(0) + f(1)b(1) \\ &\quad - j \int_0^1 f(y)b(y)\, P_{j-2}^{(1,1)}(2y-1)\, dy \Big) \end{aligned}$$

for every polynomial f of degree less than m and $x \in [0,1]$, we can proceed along the same lines of the proof of Theorem 2.1, and hence we indicate only the main steps.

Let $t > 0$, f a polynomial of degree less than m and $x, y \in [0,1]$. Then

$$\begin{aligned}
T(t)f(x) - T(t)f(y) &= (f(0)b(0) + f(1)b(1)) \left(\frac{1}{2b(x)} - \frac{1}{2b(y)}\right) \\
&+(f(1)b(1) - f(0)b(0)) \left(\frac{x-1/2}{b(x)} - \frac{y-1/2}{b(y)}\right) \\
&+ \left(\frac{1}{b(y)} - \frac{1}{b(x)}\right) \sum_{j=2}^{m} e^{-j(j-1)t/2}\, \frac{2j-1}{j-1}\, x(1-x)P_{j-2}^{(1,1)}(2x-1) \times \\
&\times \left\{(-1)^j f(0)b(0) + f(1)b(1) - j \int_0^1 f(s)b(s)\, P_{j-2}^{(1,1)}(2s-1)\, ds\right\} \\
&- \frac{1}{b(y)} \sum_{j=2}^{m} e^{-j(j-1)t/2}\, \frac{2j-1}{j-1} \times \\
&\times \left[x(1-x)P_{j-2}^{(1,1)}(2x-1) - y(1-y)P_{j-2}^{(1,1)}(2y-1)\right] \times \\
&\times \left\{(-1)^j f(0)b(0) + f(1)b(1) - j \int_0^1 f(s)b(s)\, P_{j-2}^{(1,1)}(2s-1)\, ds\right\}.
\end{aligned}$$

Thus, taking into account that $b \in C[0,1]$, $b_0 := \min_{x\in[0,1]} b(x) > 0$, we can argue as in the proof of Theorem 2.1 and obtain that, for all $x, y \in [0,1]$,

$$\begin{aligned}
|T(t)f(x) - T(t)f(y)| \quad \leq \quad & \|f\|\, \|b\,\| \left|\frac{1}{b(x)} - \frac{1}{b(y)}\right| \\
& +2\|f\|\, \|b\,\| \left|\frac{x-1/2}{b(x)} - \frac{y-1/2}{b(y)}\right| + \\
& +k\|f\|\, \|b\,\| \left|\frac{1}{b(x)} - \frac{1}{b(y)}\right| + \frac{1}{b_0} k_1 \|f\|\, |x-y|\,,
\end{aligned}$$

where the positive constants k, k_1 depends only on t. From this the immediately compactness property directly follows. ■

2.2 Bernstein-Durrmeyer operators

The Bernstein-Durrmeyer operator $M_n\colon L^2(0,1) \to L^2(0,1)$ is defined by setting

$$M_n f(x) := (n+1) \sum_{k=0}^{n} p_{k,n}(x) \int_0^1 f(t)\, p_{k,n}(t) dt \tag{2.9}$$

for every $f \in L^2(0,1)$ and $x \in [0,1]$, where

$$p_{k,n}(x) := \binom{n}{k} x^k\, (1-x)^{n-k}\,, \qquad k = 0, \ldots, n\,, \quad x \in [0,1]\,.$$

We have (see, e.g. [1, p. 336])

$$M_n(\mathbf{1})(x) = 1, \quad M_n(\mathrm{id})(x) = \frac{nx+1}{n+2}, \quad M_n(\mathrm{id}^2)(x) = \frac{n(n-1)x^2+4nx+2}{(n+2)(n+3)} \ ;$$

moreover by [4] the eigenvalues of M_n are

$$\lambda_{k,n} = \frac{n!}{(n-k)!}\frac{(n+1)!}{(n+k+1)!} \ , \qquad k = 0, \ldots, n \ ,$$

and the corresponding eigenfunctions are the Legendre polynomials

$$q_k(x) := P_k(2x-1) = P_k^{(0,0)}(2x-1) \ , \qquad k = 0, \ldots, n \ , \quad x \in [0,1] \ .$$

Notice that if $(j_n)_{n\geq 1}$ is a sequence of positive integers with

$$\lim_{n\to+\infty} \frac{j_n}{n} = t \in [0, +\infty[\ ,$$

then, for every $k \in \mathbf{N}$,

$$\lim_{n\to+\infty} (\lambda_{k,n})^{j_n} = e^{-k(k+1)t} \ . \tag{2.10}$$

Indeed, for a fixed $k \in \mathbf{N}$ and $n \geq k$, we may write

$$(\lambda_{k,n})^{j_n} = \left(\frac{n!}{n^k(n-k)!}\right)^{j_n} \left(\frac{n^k(n+1)!}{(n+k+1)!}\right)^{j_n} \ , \tag{2.11}$$

where by [3, (5.11)]

$$\lim_{n\to+\infty} \left(\frac{n!}{n^k(n-k)!}\right)^{j_n} = e^{-k(k-1)t/2}. \tag{2.12}$$

Next, let

$$y_{n,k} := \left(\frac{n^k(n+1)!}{(n+k+1)!}\right)^{j_n-nt} = \left(\frac{1}{(1+\frac{k+1}{n})(1+\frac{k}{n})\ldots(1+\frac{2}{n})}\right)^{j_n-nt} .$$

Then

$$\begin{aligned}
\log y_{n,k} &= -(j_n-nt)\left(\log(1+\frac{k+1}{n}) + \log(1+\frac{k}{n}) + \ldots \log(1+\frac{2}{n})\right) \\
&= -(j_n-nt)\left(\frac{k+1}{n} + \frac{k}{n} + \ldots \frac{2}{n} + o\left(\frac{1}{n}\right)\right) \\
&= -\left(\frac{j_n}{n} - t\right)\left(\frac{k(k+3)}{2} + o\left(\frac{1}{n}\right)\right) ;
\end{aligned}$$

hence $\lim_{n\to+\infty} \log y_{n,k} = 0$ and therefore

$$\lim_{n\to+\infty} \left(\frac{n^k(n+1)!}{(n+k+1)!}\right)^{j_n-nt} = \lim_{n\to+\infty} y_{n,k} = 1 \ .$$

On the other hand,

$$\begin{aligned}\lim_{n\to+\infty}\left(\frac{n^k(n+1)!}{(n+k+1)!}\right)^{nt} &= \lim_{n\to+\infty}\frac{1}{(1+\frac{k+1}{n})^{nt}(1+\frac{k}{n})^{nt}\dots(1+\frac{2}{n})^{nt}} \\ &= e^{-k(k+3)t/2}\ .\end{aligned}$$

By (2.11) and (2.12) it follows that

$$\lim_{n\to+\infty}(\lambda_{k,n})^{j_n} = e^{-k(k-1)t/2}\cdot e^{-k(k+3)t/2} = e^{-k(k+1)t}\ .$$

Recall that the orthonormal Legendre polynomials in $[0,1]$ have the form (see [9])

$$p_j(x) := \sqrt{\frac{2k+1}{2}}\,P_k(2x-1)\ . \tag{2.13}$$

Then the dual functionals $\mu_j\colon L^2(0,1)\to\mathbf{R}$ to p_j are defined by

$$\mu_j(f) = \int_0^1 f(t)\,p_j(t)\,dt\ ,\quad f\in L^2(0,1)\ . \tag{2.14}$$

Therefore, the Bernstein-Durrmeyer operator may be represented in the diagonal form

$$M_n(f) = \sum_{j=0}^{n}\lambda_{j,n}p_j\,\mu_j(f)\ ,\quad f\in L^2(0,1)\ . \tag{2.15}$$

Since $(p_j)_{j\in\mathbf{N}}$ is an orthonormal system, we also have

$$\begin{aligned}\|M_nf\|_2 &= \left(\sum_{j=0}^{n}\lambda_{j,n}^2|\mu_j(f)|^2\right)^{1/2} \qquad (2.16)\\ &\le \left(\max_{j=0,\dots,n}\lambda_{j,n}^2\right)^{1/2}\left(\sum_{j=0}^{n}|\mu_j(f)|^2\right)^{1/2}\le\|f\|_2\end{aligned}$$

for all $f\in L^2(0,1)$.

By induction on $k\in\mathbf{N}$ and using the identity $M_n(p_j)=\lambda_{j,n}p_j$, we easily get

$$M_n^kf = \sum_{j=0}^{n}(\lambda_{j,n})^kp_j\,\mu_j(f)\ ,\quad f\in L^2(0,1)\ . \tag{2.17}$$

Moreover, it is well-known that the differential operator associated with the Voronovskaja formula for the Bernstein-Durrmeyer operators is given by

$$Bf(x) := x(1-x)f''(x)+(1-2x)f'(x)\ ,$$

for $f\in C^2[0,1]$ (see, e.g. [4] or [11]).

The operator B generates a strongly continuous positive semigroup $(T(t))_{t\geq 0}$ on $L^2(0,1)$ on the domain

$$D(B) := \{f \in L^2(0,1) \quad | \quad f \text{ is locally absolutely continuous in }]0,1[\text{ and } x(1-x)\, f'(x) \in W_0^{1,2}(0,1)\}\ ,$$

where $W_0^{1,2}(0,1)$ denotes as usual the closure of $C_0^\infty(0,1)$ in the Sobolev space $W^{1,2}(0,1)$ (see, e.g. [2]).

According to the Trotter's theorem [10, Theorem 5.1], since the iterates of M_n satisfy $\|M_n^k\| \leq 1$, we also have the following representation

$$T(t)f = \lim_{n\to\infty} M_n^{[nt]} f$$

for all $f \in L^2(0,1)$.

We are now able to state and prove the following result.

Theorem 2.3 *The semigroup generated by the differential operator $(B, D(B))$ is analytic and immediately compact in $L^2(0,1)$.*

Proof. Let $\lambda \in \mathbf{C}$ be such that $\operatorname{Re}\lambda > 0$ and consider the resolvent operator $R(\lambda, B)$. If f is a real-valued polynomial of degree m, we have

$$R(\lambda, B)f = \lim_{n\to+\infty} \sum_{j=0}^{m} p_j \mu_j(f) \int_0^{+\infty} e^{-\lambda t} (\lambda_{j,n})^{[nt]}\, dt\ .$$

Taking account that $\lambda_{0,n} = 1$, by (2.10) and by dominated convergence theorem we obtain

$$R(\lambda, B)f = \frac{1}{\lambda}\left(p_0\mu_0(f) + \sum_{j=1}^{m} \frac{\lambda}{\lambda + j(j+1)}\, p_j\mu_j(f)\right)\ . \tag{2.18}$$

Since $\operatorname{Re}\lambda > 0$, it holds

$$\begin{aligned}\left\|\sum_{j=1}^{m} \frac{1}{\lambda + j(j+1)}\, p_j\mu_j(f)\right\|_2 &= \left(\sum_{j=1}^{m} \frac{1}{|\lambda + j(j+1)|^2}\, |\mu_j(f)|^2\right)^{1/2} \\ &\leq \left(\max_{j=1,\dots,m} \frac{1}{|\lambda + j(j+1)|^2}\right)^{1/2} \left(\sum_{j=1}^{m} |\mu_j(f)|^2\right)^{1/2} \\ &\leq \frac{1}{|\lambda|}\, \|f\|_2\end{aligned}$$

and

$$\int_0^1 |p_0\mu_0(f)|^2 dx = \int_0^1 \left|\int_0^1 f(t)dt\right|^2 dx \leq \|f\|_2^2\ .$$

Thus, we get

$$\|R(\lambda, B)f\|_2 \leq \frac{2}{|\lambda|}\|f\|_2 \ .$$

Since the space of polynomials is dense in $L^2(0,1)$, the above inequality holds for every $f \in L^2(0,1)$ and also extends to every $f \in L^2((0,1), \mathbf{C})$. This proves the analyticity of the semigroup.

Finally, we show that the semigroup $(T(t))_{t\geq 0}$ generated by the differential operator $(B, D(B))$ is also immediately compact, that is $T(t)$ is compact for all $t > 0$. Indeed, $(T(t))_{t\geq 0}$ is immediately norm continuous in $L^2(0,1)$ as it is analytic in $L^2(0,1)$ (see [5, Chapter II, Section 4.c, p. 112]). On the other hand, since the orthonormal Legendre polynomials $(p_j)_{j\in\mathbf{N}}$ form a complete system in $L^2(0,1)$, from (2.18) it follows that, for every $f \in L^2(0,1)$,

$$R(\lambda, B)f = \frac{1}{\lambda}\, p_0\, \mu_0(f) + \sum_{j=1}^{\infty} \frac{1}{\lambda + j(j+1)}\, p_j\, \mu_j(f) \ ,$$

where $\left(\frac{1}{\lambda+j(j+1)}\right)_{j\in\mathbf{N}} \in \ell^1$; hence $R(\lambda, B)$ is a nuclear and hence a compact operator. Since $(T(t))_{t\geq 0}$ is immediately norm continuous and its generator has compact resolvent, $(T(t))_{t\geq 0}$ is immediately compact (see [5, Theorem 4.29, p. 119]). ∎

Remark 2.4 The semigroup $(T(t))_{t\geq 0}$ generated by the differential operator $(B, D(B))$ is bounded analytic with angle $\pi/2$. Indeed, proceeding as in the proof of Theorem 2.3, one shows that

$$T(t)f = p_0\, \mu_0(f) + \sum_{j=1}^{\infty} e^{-j(j+1)t} p_j \mu_j(f)$$

for all $f \in L^2(0,1)$ and $t \geq 0$. Thus $(T(t))_{t\geq 0}$ extends analytically on $\Delta = \{z \in \mathbf{C} \setminus \{0\} \mid |\operatorname{Arg} z| < \pi/2\}$ by defining

$$T(z)f = p_0\, \mu_0(f) + \sum_{j=1}^{\infty} e^{-j(j+1)z} p_j \mu_j(f)$$

for $f \in L^2(0,1)$ and $z \in \Delta$; hence $\|T(z)\| \leq 1$ for all $z \in \Delta$.

Remark 2.5 Finally, with the same arguments used for Bernstein operators, we can establish similar qualitative properties for semigroups generated by suitable perturbations of the differential operator B in the space $L^2(0,1)$.

Consequently, if $b \in C^2[0,1]$ is a strictly positive function, we obtain that the differential operator

$$B_b f(x) := x(1-x)\, f''(x) + \left(1 - 2x + 2x(1-x)\, \frac{b'(x)}{b(x)}\right)\, f'(x)$$

generates an analytic and immediately compact C_0-semigroups in $L^2(0,1)$ on the domain

$$D(B_b) := \{f \in L^2(0,1) \quad | \quad f \text{ is locally absolutely continuous in }]0,1[\text{ and } x(1-x)\, f'(x) \in W_0^{1,2}(0,1)\}\ .$$

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Risk Attribution and Portfolio Performance Measurement-An Overview

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Abstract

A major problem associated with risk management is that it is very hard to identify the main source of risk taken, especially in a large and complex portfolio. This is due to the fact that the risk of individual securities in the portfolio, measured by most of the widely used risk measures such as the standard deviation or the Value-at-Risk, don't sum up to the total risk of the portfolio. Although the risk measure of beta in the Capital Asset Pricing Model seems to survive this major deficiency, it suffers too much from the controversial model it is inherently based on to become a satisfactory solution. Risk attribution is a technique of decomposing the total risk of a portfolio into smaller terms, each of which can be interpreted as the risk contribution of the corresponding subsets of the portfolio. This technique is so powerful

that it can be applied to any homogeneous risk measure in any model. We present here an overview of the methodology of risk attribution with some widely used risk measures and compare their properties under different assumptions of the payoff distributions.

AMS Subject Classification Code: *91B30*

Key Words: *Risk attribution, performance measurement, risk measures, Value-at-Risk, Conditional Value-at-Risk, Stable distributions.*

1 Introduction

The central task of risk management is to locate the main source of risk and find trading strategies to hedge risk. But the major problem is that it is very hard to identify the main source of risk taken, especially in a large and complex portfolio. This is due to the fact that the risk of individual securities, measured by most of the widely used risk measures such as standard deviation and value-at-risk, don't sum up to the total risk of the portfolio. That is, while the stand-alone risk of an individual asset is very significant, it could contribute little to the overall risk of the portfolio because of its correlations with other securities in the portfolio. It could even act as a hedge instrument that reduces the overall risk. Although the risk measure of beta in the Capital Asset Pricing Model(CAPM) survives the above detrimental shortcoming (i.e. the weighted sum of individual betas equals the portfolio beta), it suffers from the shortcomings of the model that it is based on. Furthermore, beta is neither translation-invariant nor monotonic, which are two key properties possessed by a coherent risk measure that we discuss later in section 4.2. Except for the beta, the stand-alone risk measured by other risk measures provide little information about the composition of the total risk and thus give no hint on how to hedge risk.

Risk attribution is a technique of decomposing the total risk of a portfolio into smaller units, each of which can be interpreted as the risk contribution of the corresponding subset of securities in the portfolio.[1] The methodology applies to any risk measures, as long as they are homogeneous (see definition

[1]Mina(2002) shows that the decompostion can be taken according to any arbiturary partition of the portfolio. The partition could be made according to active investment decisions, such as sector allocation and security selection. For example, the partition could be made by countries, sectors or industries.

3.2). Risk measures that have better properties than beta but are not additive are now remedied by risk attribution. The problem of identifying major source of risk is then solved. After the primary source of the risk is identified, active portfolio hedging strategies can be carried out to hedge the significant risk already taken.

It is worthwhile to mention that we assume there exits a portfolio already before the risk attribution analysis is taken. This pre-existing portfolio could be a candidate of an optimized portfolio or even an optimized portfolio. But this is counterintuitive since the optimized portfolio is supposed to be the "best" already. Why shall we bother? The answer to this question is the heart of *risk hedging*. Investors are typically risk averse and they don't feel comfortable when they find out through risk attribution that the major risk of their optimized portfolios is concentrated on one or few securities. They are willing to spend extra money on buying financial insurance such as put and call options in order to hedge their positions of major risk exposure. Furthermore, Kurth et al. (2002) note that optimal portfolios are quite rare in practice, especially in credit portfolio management. It is impossible to optimize a portfolio in one step, even if the causes of bad performance in a credit portfolio have been located. It is still so in general portfolio management because of rapid changes of market environment. Traders and portfolio managers often update their portfolios on a daily basis. Successful portfolio management is indeed a process consisting of small steps, which requires detailed risk diagnoses, namely risk attribution. The process of portfolio optimization and risk attribution can be repeatedly performed until the satisfactory result is achieved.

There are certainly more than one risk measure that are proposed and used by both academics and practitioners. There has been a debate about which risk measure is more appropriate ([3] and [4]). While the basic idea of decomposing the risk measure is the same, the methods of estimating and computing the components could differentiate a lot for different risk measures. We give a close look at these risk measures and show their properties under different assumptions of payoff distributions.

In the next section, we briefly examine beta in the Capital Asset Pricing Model developed by Sharpe and Tobin. Although beta can be a risk measure whose individual risk components sum up to the total risk, the model in which it lives in is under substantial criticism. The rejection of the questionable beta leads us to seek for new tools of risk attribution. In Section 3, under a general framework without specifying the function forms of the risk measures, the

methodology of differentiating the risk measures is presented . Different risk measures are introduced in Section 4. The methods of calculating derivatives of risk measures under different assumptions of payoff distributions follow in Section 5. The last section concludes the paper and gives light to future studies.

2 What is wrong with beta?

The Capital Asset Pricing Model (CAPM) is a corner stone of modern finance. It states that when all investors have the same expectation and the same information about the future, the expected excess return of a security is proportional to the risk, measured by beta, of the security. This simple yet powerful model can be expressed mathematically as follows:

$$E[R_i] - R_f = \beta_i(E[R_M] - R_f) \tag{1}$$

where R_i is the random return of the *ith* security, R_f is the return of the riskless security and R_M is the random return of *the market portfolio*, which is defined as the portfolio consisting of all the assets in the market. Beta β_i is defined as the ratio of the covariance between security i and the market portfolio and the variance of the market portfolio.

$$\beta_i = \frac{Cov(R_i, R_M)}{\sigma_M^2} \tag{2}$$

As so defined, beta measures the responsiveness of asset i to the movement of the market portfolio. Under the assumption that agents have homogeneous expectations and information set, all investors will combine some of the riskless asset and some of the market portfolio to form their own optimal portfolios. Since everyone investor holds the market portfolio, beta can be a measure of risk in the CAPM. The weighted sum of betas of individual securities in a portfolio equals the beta of the portfolio, i.e.

$$\beta_P = \sum_{i=1}^{N} w_i\beta_i \tag{3}$$

where w_i is the portfolio weight, i.e. the percentage of wealth invested in security i. We can identify the main source of risk by examining the values of

betas. The security with the largest weighted beta contributes most to the total risk of the portfolio. Risk attribution seems easy in the CAPM world.

However, the CAPM and beta have been criticized for their over simplicity and not being representative about the real world we live in. The famous Roll's critique(1977) asserts that the market portfolio is not observable and the only testable implication of the CAPM is the efficiency of the market portfolio. Another representative challenge is the paper by Fama and French(1992). They find some empirical evidence showing that beta and long-run average return are not correlated. While some people don't like the CAPM and beta, others still do (cf. [13] for example). But we don't attempt to give an extensive survey of the CAPM literature. The point we want to make is that beta, as well as the CAMP that it is based on, is too controversial for us to reply on, especially when there exists an alternative way to develop risk attribution technique.

3 Risk Attribution-The Framework

Risk attribution is not a new concept. The term stems from the term *"return attribution"* or *"performance attribution"*, which is a process of attributing the return of a portfolio to different factors according to active investment decisions. The purpose is to evaluate the ability of asset managers. The literature on return attribution or performance attribution started in the 60s,[2] when mutual funds and pension funds were hotly debated. Whereas the literature on risk attribution, which is closely related to but different from return attribution, didn't start until the mid 90's. Risk attribution differs from return attribution in two major aspects. First, as is clear from their names, the decomposing objectives are risk measures for the former and returns for the latter. Second, the latter uses historical data and thus is ex-post while the former is an ex-ante analysis (cf. [21]).

The general framework of decomposing risk measures was first introduced by Litterman (1996), who uses the fact that the volatility, defined as the standard deviation, of a portfolio is linear in position size. His finding is generalized to risk measures that possess this property, which we call later *homogeneity.* We shall formally define the general measure of risk.

We assume there are N financial assets in the economy. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{G}$ be a set of random variables which are $\mathcal{F}$-

[2] See Fama (1972) for a short review.

measurable. The $N \times 1$ random vector R is the return vector in which the ith element R_i is the random return of asset $i,\ i = 1, 2, \ldots, N$. Let the $N \times 1$ vector $m \in \mathbb{R}^N$ be the vector of portfolio positions where the ith element m_i is the amount of money invested in asset $i,\ i = 1, 2, \ldots, N$. A portfolio is represented by the vector m and the portfolio random payoff is $X = m'R \in \mathcal{G}$. All returns belong to some time interval Δt and all positions are assumed to be at the beginning of the time interval.

Definition 3.1 *A **risk measure** is a mapping $\rho : \mathcal{G} \to \mathbb{R}$.*

Definition 3.2 *A risk measure is **homogeneous of degree** τ if $\rho(kX) = k^\tau \rho(X)$, for $X \in \mathcal{G}$, $kX \in \mathcal{G}$ and $k > 0$.*[3]

We are more interested in risk measures which is homogeneous of degree one because this is one of the properties possessed by the class of *coherent measures*, which we define in the next section. In Litterman's framework, homogeneity of degree one plays a central role in decomposing risk measures into meaningful components. When all position sizes are multiplied by a common factor $k > 0$, the overall portfolio risk is also multiplied by this common factor. As we can see from the following proposition that each component can be interpreted as the marginal risk contribution of each individual security or a subset of securities in the portfolio from small changes in the corresponding portfolio position sizes.

Proposition 3.3 *(Euler's Theorem) Let ρ be a homogeneous risk measure of degree τ. If ρ is partially differentiable with respect to $m_i,\ i = 1, \ldots, N$,*[4] *then*

$$\rho(X) = \frac{1}{\tau}(m_1 \frac{\partial \rho(X)}{\partial m_1} + \cdots + m_N \frac{\partial \rho(X)}{\partial m_N}) \tag{4}$$

Proof. *Consider a mapping $\mu : \mathbb{R}^N \to \mathcal{G}$ and $\mu(m) = m'R$, for $m \in \mathbb{R}^N$. Then $\rho(X) = \rho \circ \mu(m)$, where $\rho \circ \mu$ is a composite mapping: $\mathbb{R}^N \to \mathcal{G} \to \mathbb{R}$. Homogeneity of degree τ implies that for $k > 0$,*

$$\rho(kX) = \rho \circ \mu(km) = k^\tau \rho \circ \mu(m) = k^\tau \rho(X) \tag{5}$$

[3]Some authors use the term *positive homogeneity* to refer to the case when $\tau = 1$ i.e. the risk measure is homogeneous of degree one.

[4]Tasche (1999) has a slightly more general assumption. He shows that if the risk measure is $\tau-$homogeneous, continuous and partially differentiable with respect to $m_i,\ i = 2, \ldots, N$, then it is also differentiable with respect to m_1.

Taking the first-order derivative with respect to k to equation (5), we have

$$\begin{aligned}\frac{d\rho(kX)}{dk} &= \frac{\partial\rho\circ\mu(km)}{\partial km_1}m_1+\cdots+\frac{\partial\rho\circ\mu(km)}{\partial km_N}m_N\\ &= k^{\tau-1}(\frac{\partial\rho(X)}{\partial m_1}m_1+\cdots+\frac{\partial\rho(X)}{\partial m_N}m_N)=\tau k^{\tau-1}\rho(X)\end{aligned}$$

Deviding both sides of the above equation by $k^{\tau-1}$ gives the result. ■

In particular, we are more interested in the case when $\tau = 1$, since most widely used risk measures fall into this category. The above proposition (known as the Euler's Theorem) is fundamental in risk attribution. It facilitates identifying the main source of risk in a portfolio. Each component $m_i\frac{\partial\rho(X)}{\partial m_i}$, termed as the *risk contribution of asset i*, is the amount of risk contributed to the total risk by investing m_i in asset i. The sum of risk contributions over all securities equals the total risk. If we rescale every risk contribution term by $\frac{1}{\rho(X)}$, we get the percentage of the total risk contributed by the corresponding asset.

The term $\frac{\partial\rho(X)}{\partial m_i}$ is called *marginal risk* which represents the marginal impact on the overall risk from a small change in the position size of security i, keeping all other positions fixed. If the sign of marginal risk of one asset is positive, increasing the position size of the asset by a small amount will increase the total risk; If the sign is negative, increasing the position size of the asset by a small amount will reduce the total risk. Thus the securities with negative marginal risk behave as hedging instruments.

But there is one important limitation of this approach ([17]). The decomposition process is only a marginal analysis, which implies that only small changes in position sizes make the risk contribution terms more meaningful. For example, if the risk contribution of security a–in a portfolio consisting of only two securities–is twice that of security b, then a small increase in the position of security a will increase the total risk twice as much as the one caused by the same amount of increase in the position of security b. However, it doesn't imply that removing security a from the portfolio will reduce the overall risk by 2/3. In fact, the marginal risk and the total risk will both change as the position size of security a changes. This is because of the definition of marginal risk. *Only if* an increase (say ε_1) in asset one's position size is small enough, the additional risk, or *incremental risk,*[5] can be approx-

[5] We define the incremental risk as the difference between the total risk after changing

imated by $\frac{\partial\rho(X)}{\partial m_1}\varepsilon_1$.[6] The larger ε_1 is, the poorer the approximation would be. Removing asset one entirely represents a large change in the position size and thus the approach is not suitable in this situation.

This limitation casts doubt on the philosophy of risk attribution. The main questions are:

1. Is each of the decomposed terms in (4) really an appropriate representation of the risk contribution of each individual asset?

2. If dropping the asset with most risk contribution won't help, what do we do in order to reduce the overall risk?

We don't answer the second question for now. The answer may be found in examining the interaction and relationship between risk attribution and portfolio optimization, which is a topic for future studies.

To answer the first question, two author's work are worth mentioning. Tasche (1999) shows that under a general definition of suitability, the only representation appropriate for performance measurement is the first order partial derivative of the risk measure with respect to position size, which is exactly the marginal risk we define. Denault (2001) applies game theory to justify the use of partial derivatives of risk measures as a measure of risk contribution. We hereby briefly discuss the approach by Tasche ([29]).

The approach is more like an axiomatic one, that is, to first define the universally accepted and self-evident properties or principles the object possesses, then to look for "candidates" which satisfy the pre-determined criterion.[7] In his attempt to define the criterion, Tasche makes use of the concept of *"Return on Risk-Adjusted Capital"*(RORAC), which has been used in allocating banks' capital. RORAC is defined as the ratio between some certain measure of profit and the bank's internal measure of capital at risk.([20]) In our notation, the *portfolio's RORAC* should be $\frac{E[mR]}{\rho(X)-E[m'R]}$, where $E[mR]$ is the expected payoff of the portfolio and $\rho(X) - E[m'R]$ is the *economic capital*, which is the amount of capital needed to prevent solvency at some confidence level. For every unit of investment (i.e. the position size equals 1), the RORAC of an individual asset i (or *per-unit RORAC of asset i*) can be

the composition of a portfolio and the total risk before the change. Note that some authors (cf. [21]) define the risk contribution $m_i\frac{\partial\rho(X)}{\partial m_i}$ as the incremental risk.

[6] A Taylor series expansion can be performed to yield this result ([11]).

[7] Other examples of the axiomatic approach can be found in [4] and [6].

denoted by $\frac{E[R_i]}{\rho(R_i)-E[R_i]}$, where $\rho(R_i)$ is the risk measure of asset i per unit of position i. The RORAC is very similar to the Sharpe ratio, which measures the return performance per unit of risk. If the RORAC of capital A is greater than that of capital B, then capital A gives a higher return per unit of risk than B and thus has a better performance than B; and vice versa. Tasche defines that the measure suitable for performance measurement should satisfy the following conditions:

i) If, for every unit of investment, the amount $\frac{E[R_i]}{a_i(X)-E[R_i]}$[8] is greater than that of the entire portfolio, then investing a little more in asset i should enhance the performance (measured by RORAC) of the entire portfolio, and reducing the amount invested in asset i should decrease the RORAC of the entire portfolio; In mathematical expressions, this is equivalent to

$$\begin{aligned} \frac{E[R_i]}{a_i(X)-E[R_i]} &> \frac{E[m'R]}{\rho(X)-E[m'R]} \\ \Rightarrow \frac{E[m'R+m_i^\varepsilon R_i]}{\rho(X+m_i^\varepsilon R_i)-E[m'R+m_i^\varepsilon R_i]} &> \frac{E[m'R]}{\rho(X)-E[m'R]} > \\ & \frac{E[m'R-m_i^\varepsilon R_i]}{\rho(X-m_i^\varepsilon R_i)-E[m'R-m_i^\varepsilon R_i]} \end{aligned}$$

where $a_i(X)$ is the candidate measure of suitable risk contribution of asset i, $E[R_i]$ is the expected payoff per unit of position i and $0 < m_i^\varepsilon < \varepsilon$, for some small $\varepsilon > 0$.

ii) If, for every unit of investment, the amount $\frac{E[R_i]}{a_i(X)-E[R_i]}$ is smaller than that of the entire portfolio, then investing a little more in asset i should decrease the performance of the entire portfolio, and reducing the amount invested in asset i should enhance the RORAC of the entire portfolio; In mathematical expressions, this is equivalent to

$$\begin{aligned} \frac{E[R_i]}{a_i(X)-E[R_i]} &< \frac{E[m'R]}{\rho(X)-E[m'R]} \\ \Rightarrow \frac{E[m'R+m_i^\varepsilon R_i]}{\rho(X+m_i^\varepsilon R_i)-E[m'R+m_i^\varepsilon R_i]} &< \frac{E[m'R]}{\rho(X)-E[m'R]} < \\ & \frac{E[m'R-m_i^\varepsilon R_i]}{\rho(X-m_i^\varepsilon R_i)-E[m'R-m_i^\varepsilon R_i]} \end{aligned}$$

[8] Note that the initial risk measure $\rho(R_i)$ in the per-unit RORAC of asset i is replaced by the candidate of performance measure $a_i(X)$ of asset i.

where $a_i(X)$, $E[R_i]$ and m_i^{ε} are defined in the same way as in 1.

Tasche shows in Theorem 4.4 ([29]) that the *only* function form fulfill the above requirements is $\frac{\partial\rho(X)}{\partial m_i}$.

We note that the limitation of his criterion is that the use of RORAC is more appropriate to banks than to other financial institutions. The notion of the economic capital $\rho(X) - E[m'R]$ makes sense only when the risk measure $\rho(X)$ represents a loss threshhold at some certain level, such as the Value-at-Risk and the Expected Shortfall which we define in the next section. Nevertheless, the use of $\frac{\partial\rho(X)}{\partial m_i}$ as a measure for risk contribution is justified in the sense that it indicates how the global performance changes if there is a little change locally, given the local performance relationship with the overall portfolio.

4 Risk Measures

We examine three major risk measures, which are the *Standard Deviation (and its variants),* the *Value-at-Risk* and the *Expected Shortfall.* Their strength and weakness in measuring risk are compared. The criterion of good risk measures, namely *coherent risk measures* are reviewed.

4.1 Standard Deviation and Its Variants

Following Markowitz ([19]), scholars and practitioners has been taking the standard deviation as a "standard" risk measure for decades. Its most popular form of practical use is called the *Tracking error,* which is defined as the standard deviation (also known as the *volatility*) of the excess return (or payoff) of a portfolio relative to a benchmark[9]. Despite its appealing feature of computational ease, the standard deviation has been criticized for its inefficiency of representing risk. The inherent flaw stems from the definition of the standard deviation: both the fluctuations above the mean and below the mean are taken as contributions to risk. It implies that a rational investor would hate the potential gain to the same degree as the potential loss, if the standard deviation were used as the risk measure when he optimizes his

[9]Tracking error is sometimes defined as the difference of expected returns between a portfolio and a benchmark. Here we define it as the risk measure of the standard deviation associated with the excess return, because it is widely accepted by practioners.

portfolio. Furthermore, the standard deviation underestimates the tail risk of the payoff distribution, especially when the distribution is nonsymmetric.

To remedy the deficiency of the standard deviation, Markowitz ([19]) proposed a variant of the standard deviation, which emphasizes on the loss part of the distribution. The general form is called *the lower semiα-moment*. It is defined as follows:

$$\rho(X) = \rho(m'R) = \sqrt[\alpha]{E[((m'R - E(m'R))^-)^\alpha]} \tag{6}$$

where $(m'R - E(m'R))^- = \begin{cases} -(m'R - E(m'R)) \text{ if } m'R - E(m'R) < 0 \\ 0 \text{ if } m'R - E(m'R) \geq 0 \end{cases}$.
Note that when $\alpha = 2$, $\rho(m'R) = \sqrt{E[((m'R - E(m'R))^-)^2]}$ is called the lower semi-standard deviation, which was proposed by Markowitz.

4.2 Value-at-Risk (VaR)

Value-at-risk, or VaR for short, has been widely accepted as a risk measure in the last decade and has been frequently written into industrial regulations(see [15] for an overview). The main reason is because it is conceptually easy. It is defined as the minimum level of losses at a confidence level of solvency of $1-\alpha$. That is, VaR can be interpreted as the minimum amount of capital needed as reserve in order to prevent insolvency which happens with probability α.

Definition 4.1 *The VaR at confidence level $(1-\alpha)$*[10] *is defined as the negative of the lower α-quantile of the gain/loss distribution, where $\alpha \in (0, 1)$. i.e.*

$$VaR_\alpha = VaR_\alpha(X) = -q_\alpha(X) = -\inf\{x | P(X \leq x) \geq \alpha\} \tag{7}$$

where $P(\cdot)$ is the probability measure.

An alternative definition of VaR is that $VaR_\alpha(X) = E[X] - q_\alpha(X)$, which is the difference between the expected value of X and the lower α-quantile of X. This relative form of VaR is already used in the performance measurement of the Sharpe ratio in the last section.

Before we introduce the properties of VaR and evaluate how good or bad it is, we have to first introduce the judging rules. Four criterion have been proposed by Artzener et al. (1999).

[10] Typically, α takes the values such as 1%, 5% and 10%.

Axiom 4.2 *A risk measure $\rho : \mathcal{G} \rightarrow \mathbb{R}$ is called a* ***coherent risk measure*** *if and only if it satisfies the following properties:*

a Positive homogeneity. (See Definition 3.2)

b Monotonicity: $X \in \mathcal{G},\ X \leq 0 \Rightarrow \rho(X) > 0.$

c Translation invariance: $X \in \mathcal{G},\ c \in \mathbb{R} \Rightarrow \rho(X + c) = \rho(X) - c$

d Subadditivity: $X, Y \in \mathcal{G},\ X + Y \in \mathcal{G} \Rightarrow \rho(X + Y) \leq \rho(X) + \rho(Y).$

Positive homogeneity makes sense because of liquidity concerns. When all positions are increased by a multiple, risk is also increased by the same multiple because it's getting harder to liquidate larger positions. For monotonicity, it requires that the risk measure should give a "negative" message when the financial asset has a sure loss. The translation invariance property implies that the risk-free asset should reduce the amount of risk by exactly the worth of the risk-free asset. The subadditivity is important because it represents the diversification effect. One can argue that a risk measure without this property may lead to counterintuitive and unrealistic results.[11]

VaR satisfies property a-c but in general fails to satisfy the subadditivity[12], which has been heavily criticized. Another pitfall of VaR is that it only provides a minimum bound for losses and thus ignores any huge potential loss beyond that level. VaR could encourage individual traders to take more unnecessary risk that could expose brokerage firms to potentially huge losses. In the portfolio optimization context, VaR is also under criticism because it is not convex in some cases and may lead to serious problems when being used as a constraint. The following example shows that VaR is not sub-additive(see also [30] for another example).

Example 4.3 *Consider a call(with payoff X) and a put option(with payoff Y) that are both far out-of-money, written by two independent traders. Assume that each individual position leads to a loss in the interval [-4,-2] with probability 3%, i.e. $P(X < 0) = P(Y < 0) = 3\% = p$ and a gain in the*

[11] For examle(cf.[4]), an investor could be encouraged to split his or her account into two in order to meet the lower margin requirement; a firm may want to break up into two in order to meet a capital requirement which they would not be able to meet otherwise.

[12] Note that only under the assumption of elliptical distributions is VaR sub-additive([7]). In particular, VaR is sub-additive when $\alpha < .5$ under the Gaussian assumption(cf. [4]).

interval [1,2] with probability 97%. Thus there is no risk at 5% for each position. But the firm which the two traders belong to may have some loss at 5% level because the probability of loss is now

$$P(X+Y<0) = \sum_{i=1}^{2} \binom{2}{i} p^i (1-p)^{2-i} = 1-(1-p)^2 = 5.91\%$$

Therefore $VaR_{5\%}(X+Y) > VaR_{5\%}(X) + VaR_{5\%}(Y) = 0$.

4.3 Conditional Value-at-Risk or Expected Shortfall

While VaR has gained a lot of attention during the late nineties and early this century, that fact that it is not a coherent risk measure casts doubt on any application of VaR. Researchers start looking for alternatives to VaR. A coherent measure, *conditional value-at-risk*(CVaR) or *expected shortfall*(ES) was introduced. Similar concepts were introduced in names of mean excess loss, mean shortfall, worse conditional expectation, tail conditional expectation or tail VaR. The definition varies across different writers. Acerbi and Tasche (2002) clarify all the ambiguity of definitions of the VaR and the CVaR and show the equivalence of the CVaR and the expected shortfall. At the same time independently, Rockafellar and Uryasev (2002) also show the equivalence and generalize their finding in the previous paper to general loss distributions, which incorporate discreteness or discontinuity.

Definition 4.4 *Suppose* $E[X^-] < \infty$*, the expected shortfall at the level* α *of* X *is defined as*

$$\begin{aligned} ES_\alpha &= ES_\alpha(X) \\ &= -\frac{1}{\alpha}\{E[X\mathbf{1}_{\{X\le -VaR_\alpha\}}] - VaR_\alpha(\alpha - P[X \le -VaR_\alpha])\} \\ &= -\frac{1}{\alpha}\{E[X\mathbf{1}_{\{X\le q_\alpha(X)\}}] + q_\alpha(X)(\alpha - P[X \le q_\alpha(X)])\} \end{aligned} \tag{8}$$

The expected shortfall can be interpreted as the mean of the α-tail of the loss distribution. Rockafellar and Uryasev(2002) define the conditional value-at-risk(CVaR) based on a rescaled probability distribution. Proposition 6 in Rockafellar and Uryasev(2002) confirms that the CVaR is essentially the same as the ES. The subtleness in the definition of ES becomes especially important when the loss distribution has a jump at the point of VaR, which is usually

the case in practice. Two cases of jump(or discontinuity) and discreteness of the loss distribution, which are due to Rockafellar and Uryasev (2002), are illustrated in Figure 1 and Figure 2, respectively.

If the loss distribution is continuous, then $\alpha = P[X \leq -VaR_\alpha]$ and the expected shortfall defined above will reduce to

$$ES_\alpha(X) = -E[X|X \leq -VaR_\alpha]$$

which coincides with the tail conditional expectation defined in Artzener et al. (1999). It is worth mentioning that they show that the tail conditional expectation is generally not subadditive thus not coherent (see also [2]).

We now show that the expected shortfall is a coherent risk measure.

Proposition 4.5 *The expected shortfall(or conditional value-at-risk) defined as (8) satisfies the axiom of coherent risk measures.*
Proof.

i) *Positive homogeneity:*

$$\begin{aligned} & ES_\alpha(tX) \\ = & -\frac{1}{\alpha}\{E[tX\mathbf{1}_{\{tX \leq -tVaR_\alpha(X)\}}] - tVaR_\alpha(X)(\alpha - P[tX \leq -tVaR_\alpha(X)])\} \\ = & tES_\alpha(X) \end{aligned}$$

ii) *Monotonicity:*

$$\begin{aligned} X &\leq 0, VaR_\alpha(X) > 0 \Rightarrow \\ ES_\alpha(X) &= -\frac{1}{\alpha}\{\underbrace{E[X\mathbf{1}_{\{X \leq -VaR_\alpha\}}]}_{\leq 0} - VaR_\alpha(\underbrace{\alpha - P[X \leq -VaR_\alpha]}_{=\alpha-1<0})\} \\ &> 0 \end{aligned}$$

iii) *Translation invariance:*

$$\begin{aligned} & ES_\alpha(X+c) \\ = & -\frac{1}{\alpha}\{E[X\mathbf{1}_{\{X \leq -VaR_\alpha\}}] + cP[X \leq -VaR_\alpha] \\ & -(VaR_\alpha - c)(\alpha - P[X \leq -VaR_\alpha])\} \\ = & -\frac{1}{\alpha}\{E[X\mathbf{1}_{\{X \leq -VaR_\alpha\}}] - VaR_\alpha(\alpha - P[X \leq -VaR_\alpha])\} - c \\ = & ES_\alpha(X) - c \end{aligned}$$

iv) *Subadditivity: This proof is based on Acerbi et. al. (2001). They use an indicator function as follows:*

$$1^{\alpha}_{\{X\le x\}} = \begin{cases} 1_{\{X\le x\}} & if\ P[X=x]=0 \\ 1_{\{X\le x\}} + \frac{\alpha - P[X\le x]}{P[X=x]} 1_{\{X=x\}} & if\ P[X=x]>0 \end{cases}$$

It is easy to see that

$$E[1^{\alpha}_{\{X\le x\}}] = \alpha \tag{9}$$

$$1^{\alpha}_{\{X\le q_\alpha\}} \in [0,1] \tag{10}$$

$$\frac{1}{\alpha} E[X 1^{\alpha}_{\{X\le q_\alpha\}}] = -ES_\alpha(X) \tag{11}$$

We want to show that $ES_\alpha(X+Y) \le ES_\alpha(X) + ES_\alpha(Y)$. *By 9, 10 and 11*

$$\begin{aligned}
& ES_\alpha(X) + ES_\alpha(Y) - ES_\alpha(X+Y) \\
= & \frac{1}{\alpha} E[(X+Y)1^{\alpha}_{\{X+Y\le q_\alpha(X+Y)\}} - X1^{\alpha}_{\{X\le q_\alpha(X)\}} - Y1^{\alpha}_{\{Y\le q_\alpha(Y)\}}] \\
= & \frac{1}{\alpha} E[X(1^{\alpha}_{\{X+Y\le q_\alpha(X+Y)\}} - 1^{\alpha}_{\{X\le q_\alpha(X)\}}) + \\
& Y(1^{\alpha}_{\{X+Y\le q_\alpha(X+Y)\}} - 1^{\alpha}_{\{Y\le q_\alpha(Y)\}})] \\
\ge & \frac{1}{\alpha}(q_\alpha(X) \underbrace{E[1^{\alpha}_{\{X+Y\le q_\alpha(X+Y)\}} - 1^{\alpha}_{\{X\le q_\alpha(X)\}}]}_{0} + \\
& q_\alpha(Y) \underbrace{E[1^{\alpha}_{\{X+Y\le q_\alpha(X+Y)\}} - 1^{\alpha}_{\{X\le q_\alpha(X)\}}]}_{0}) \\
= & 0
\end{aligned}$$

■

Pfulg(2000) proves that CVaR is coherent by using a different definition of CVaR, which can be represented by an optimization problem(see also [25]).

5 Derivatives of Risk Measures

We are now ready to go one step further to the core of risk attribution analysis, namely calculating the first order partial derivatives of risk measures

with respect to positions (recall from (4)). The task is not easy because the objective functions of differentiation of VaR and $CVaR$ are probability functions or quantiles. We introduce here the main results associated with the derivatives.

5.1 Tracking Error

5.1.1 Gaussian Approach

The tracking error is defined as the standard deviation of the excess return(or payoff) of a portfolio relative to a benchmark(see the footnote in section 3.1). It is a well-established result that the standard deviation is differentiable. By assuming Gaussian distributions, Garman (1996, 1997) derives the close form formula for the derivative of VaR[13] from the variance-covariance matrix . Mina (2002) implements the methodology to perform risk attribution, which incorporates the feature of institutional portfolio decision making process in financial institutions.

We first assume Gaussian distributions. Denote by $b = (b_i)_{i=1}^N$the positions of a benchmark. Let $w = (w_i)_{i=1}^N = (m_i - b_i)_{i=1}^N = m - b$ be the excess positions(also called "bet") relative to the benchmark. Then $w'R$ is the excess payoff of the portfolio relative to the benchmark. Let Ω be the variance-covariance matrix of the returns $(r_i)_{i=1}^N$. Then the tracking error is

$$TE = \sqrt{w'\Omega w} \tag{12}$$

The first order derivative with respect to w is

$$\frac{\partial TE}{\partial w} = \frac{\Omega w}{\sqrt{w'\Omega w}} = \nabla \tag{13}$$

which is an $N \times 1$ vector. Therefore the risk contribution of the bet on asset i can be written as

$$w_i \nabla_i = w_i (\frac{\Omega w}{\sqrt{w'\Omega w}})_i \tag{14}$$

The convenience of equation (13) is that we can now play with any arbitrary partition of the portfolio so that the risk contribution of a subset of the

[13]One can thus derive the derivative of the standard deviation from the one of VaR, because under normal distributions(more generally, under elliptical distributions), VaR is a linear function of the standard deviation.

portfolio can be calculated as the inner product of ∇ and the corresponding vector of bets. For example, the portfolio can be sorted by industries $I_1, \ldots, I_n$, which are mutually exclusive and jointly exhaustive. The risk contribution of industry I_j is then $\phi_j' \nabla$, where $\phi_j = w \otimes \mathbf{1}_{\{i \in I_j\}}$ and $\mathbf{1}_{\{i \in I_j\}}$ is an $N \times 1$ vector whose's ith elements is one if $i \in I_j$ and zero otherwise, for all $i = 1, \ldots, N$. We can further determine the risk contribution of different sectors in industry I_j in a similar way.

5.1.2 Spherical and Elliptical Distributions

The Gaussian distributions can be generalized to the spherical or more generally, the elliptical distributions so that the tracking error can be still calculated in terms of the variance-covariance matrix. We briefly summarize the facts about spherical and elliptical distributions. See Embrechts et al. (2001) for details.

A random vector R has a *spherical distribution* if for every orthogonal map $U \in \mathbb{R}^{N \times N}$(i.e. $U^T U = U U^T = I_N$, where I_N is the N-dimentional identity matrix), $UX \stackrel{d}{=} X$.The definition implies that the distribution of a spherical random variable is invariant to rotation of the coordinates. The characteristic function of R is $\Phi_R(\theta) = E[\exp(i\theta' R)] = \phi(\theta'\theta) = \phi(\theta_1^2 + \cdots + \theta_N^2)$ for some function $\phi : \mathbb{R}^+ \to \mathbb{R}$, which is called the *characteristic generator* of the spherical distribution and we denote $R\tilde{\ } S_N(\phi)$.Examples of the spherical distributions include the Gaussian distributions, student-t distributions, logistic distributions and etc. The random vector R is spherically distributed($R\tilde{\ } S_N(\phi)$) if and only if there exists a positive random variable D such that

$$R \stackrel{d}{=} D \cdot U \tag{15}$$

where U is a uniformly distributed random vector on the unit hypersphere(or sphere) $S_N = \{s \in \mathbb{R}^N | \, \|s\| = 1\}$.

While the spherical distributions generalize the Gaussian family to the family of symmetrically distributed and uncorrelated random vectors with zero mean, the elliptical distributions are the affine transformation of the spherical distributions. They are defined as follows:

Definition 5.1 *A random vector R has an elliptical distribution, denoted by $R\tilde{\ } E_N(\mu, \Sigma, \phi)$, if there exist $X\tilde{\ } S_K(\phi)$, an $N \times K$ matrix A and $\mu \in \mathbb{R}^N$ such that*

$$R \stackrel{d}{=} AX + \mu$$

where $\Sigma = A^T A$ *is a* $K \times K$ *matrix.*

The characteristic function is

$$\begin{aligned} \Phi_R(\theta) &= E[\exp(i\theta'(AX+\mu))] && (16) \\ &= \exp(i\theta'\mu)E[i\theta' AX] = \exp(i\theta'\mu)\phi(\theta^T\Sigma\theta)) && (17) \end{aligned}$$

Thus the characteristic function of $R-\mu$ is $\Phi_{R-\mu}(\theta) = \phi(\theta^T\Sigma\theta)$. Note that the class of elliptical distributions includes the class of spherical distributions. We have $S_N(\phi) = E_N(0, I_N, \phi)$.

The elliptical representation $E_N(\mu, \Sigma, \phi)$ is not unique for the distribution of R. For $R\tilde{\ }E_N(\mu,\Sigma,\phi) = E_N(\tilde{\mu},\tilde{\Sigma},\tilde{\phi})$, we have $\tilde{\mu} = \mu$ and there exists a constant c such that $\tilde{\Sigma} = c\Sigma$ and $\tilde{\phi}(u) = \phi(\frac{u}{c})$. One can choose $\phi(u)$ such that $\Sigma = Cov(R)$, which is the variance-covariance matrix of R. Suppose $R\tilde{\ }E_N(\mu,\Sigma,\phi)$ and $R \stackrel{d}{=} AX + \mu$, where $X\tilde{\ }S_K(\phi)$. By (15), $X \stackrel{d}{=} D \cdot U$. Then we have $R \stackrel{d}{=} ADU + \mu$. It follows that $E[R] = \mu$ and $Cov[R] = AA^T E[D^2]/N = \Sigma E[D^2]/N$ since $Cov[U] = I_N/N$. So the characteristic generator can be chosen as $\tilde{\phi}(u) = \phi(u/c)$ such that $\tilde{\Sigma} = Cov(R)$, where $c = N/E[D^2]$. Therefore, the elliptical distribution can be characterized by its mean, variance-covariance matrix and its characteristic generator.

Just like the Gaussian distributions, the elliptical class preserves the property that any affine transformation of an elliptically distributed random vector is also elliptical with the same characteristic generator ϕ. That is, if $R\ \tilde{\ }E_N(\mu,\Sigma,\phi)$, $B \in \mathbb{R}^{K\times N}$ and $b \in \mathbb{R}^N$ then $BR + b\tilde{\ }E_K(B\mu + b, B\Sigma B^T, \phi)$.

Applying these results to the portfolio excess payoff $Y = w'R$, we have $X\tilde{\ }E_1(w'\mu, w'\Sigma w, \phi)$. The tracking error is again $TE = \sqrt{w'\Sigma w}$. The derivative of the tracking error is then similar to the one under the Gaussian case derived in section (5.1.1). We can see that the variance-covariance matrix, under elliptical distributions, plays the same important role of measuring dependence of random variables, as in the Gaussian case. That is why the tracking error can still be express in terms of the variance-covariance matrix.

5.1.3 Stable Approach

It is well-known that portfolio returns don't follow normal distributions. The early work of Mandelbrot (1963) and Fama (1965) built the framework of using *stable* distributions to model financial data. The excessively peaked,

heavy-tailed and asymmetric nature of the return distribution made the authors reject the Gaussian hypothesis in favor of the stable distributions, which can incorporate excess kurtosis, fat tails, and skewness.

The class of all stable distributions can be described by four parameters: $(\alpha, \beta, \mu, \sigma)$.The parameter α is the index of stability and must satisfy $0 < \alpha \leq 2$. When $\alpha = 2$ we have the Gaussian distributions. The term *stable Paretian* distributions is to exclude the case of Gaussian distributions ($\alpha = 2$) from the general case. The parameter β, representing skewness of the distribution, is within the range $[-1, 1]$. If $\beta = 0$, the distribution is symmetric; If $\beta > 0$, the distribution is skewed to the right and to the left if $\beta < 0$. The location is described by μ and σ is the scale parameter, which measures the dispersion of the distribution corresponding to the standard deviation in Gaussian distributions.

Formally, a random variable X is stable (Paretian stable, α-stable) distributed if for any $a > 0$ and $b > 0$ there exists some constant $c > 0$ and $d \in \mathbb{R}$ such that $aX_1 + bX_2 \stackrel{d}{=} cX + d$, where X_1 and X_2 are independent copies of X. The stable distributions usually don't have explicit forms of distribution functions or density functions. But they are described by explicit characteristic functions derived through the Fourier transformation. So the alternative definition of a stable random variable X is that for $\alpha \in (0, 2]$, $\beta \in [-1, 1]$ and $\mu \in \mathbb{R}$, X has the characteristic function of the following form:

$$\Phi_X(t) = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha (1 - i\beta sign(t) \tan \frac{\pi\alpha}{2} + i\mu t)\} & if\ \alpha \neq 1 \\ \exp\{-\sigma |t| (1 + i\beta sign(t) \ln |t| + i\mu t)\} & if\ \alpha = 1 \end{cases} \tag{18}$$

Then the stable random variable X is denoted by $X\tilde{}S_\alpha(\sigma, \beta, \mu)$.In particular, when both the skewness and location parameters β and μ are zero, X is said to be symmetric $\alpha-$stable and denoted by $X\tilde{}S\alpha S$.

A random *vector* R of dimension N is multivariate stable distributed if for any $a > 0$ and $b > 0$ there exists some constant $c > 0$ and a vector $D \in \mathbb{R}^N$ such that $aR_1 + bR_2 \stackrel{d}{=} cR + D$, where R_1 and R_2 are independent copies of R.The characteristic function now is

$$\Phi_R(\theta) = \begin{cases} \exp\{-\int_{S_N} |\theta^T s|^\alpha (1 - isign(\theta^T s) \tan \frac{\pi\alpha}{2}) \Gamma(ds) + i\theta^T \mu\} & if\ \alpha \neq 1 \\ \exp\{-\int_{S_N} |\theta^T s|^\alpha (1 - i\frac{2}{\pi} sign(\theta^T s) \ln |\theta^T s|) \Gamma(ds) + i\theta^T \mu\} & if\ \alpha = 1 \end{cases} \tag{19}$$

where $\theta \in \mathbb{R}^N$, Γ is a bounded nonnegative measure(also called a *spectral measure*) on the unit sphere $S_N = \{s \in \mathbb{R}^N | \, \|s\| = 1\}$ and μ is the shift vector.

In particular, if $R\tilde{}S_\alpha S$, we have $\Phi_R(\theta) = \exp\{-\int_{S_N} |\theta^T s|^\alpha \Gamma(ds)\}$. For an in-depth coverage of properties and applications of stable distributions, we refer to Samorodnitsky and Taqqu (1994) and also Rachev and Mittnik (2000).

As far as risk attribution is concerned, we want to first express the portfolio risk under the stable assumption and then differentiate the measure with respect to the portfolio weight. If the return vector R is multivariate stable Paretian distributed($0 < \alpha < 2$), then all linear combinations of the components of R are stable with the same index α. For $w \in \mathbb{R}^N$, defined as the difference between the portfolio positions and the benchmark positions, the portfolio gain $Y = w'R = \sum_{i=1}^N w_i R_i$ is $S_\alpha(\sigma_Y,\ \beta_Y,\ \mu_Y)$. It can be shown that the scale parameter of $S_\alpha(\sigma_Y,\ \beta_Y,\ \mu_Y)$ is

$$\sigma_Y = (\int_{S_N} |w's|^\alpha \Gamma(ds))^{\frac{1}{\alpha}} \tag{20}$$

which is the analog of standard deviation in the Gaussian distribution. This implies that σ_Y is the tracking error under the assumption that asset returns are multivariate stable Paretian distributed. Thus we can use (20) as the measure of portfolio risk. Similarly, the term $\sigma_Y^\alpha = \int_{S_N} |w's|^\alpha \Gamma(ds)$ is the *variation* of the stable Paretian distribution, which is the analog of the variance.

The derivatives of σ_Y with respect to w_i can be calculated for all i.

$$\frac{\partial \sigma_Y}{\partial w_i} = \frac{1}{\alpha}(\int_{S_N} |w's|^\alpha \Gamma(ds))^{\frac{1}{\alpha}-1}(\int_{S_N} \alpha |w's|^{\alpha-1}|s_i| \Gamma(ds)) \tag{21}$$

As a special case of stable distributions, as well as a special case of the elliptical distributions, we look at the *sub-Gaussian* $S_\alpha S$ random vector. A random vector R is sub-Gaussian $S_\alpha S$ distributed if and only if it has the characteristic function $\Phi_Z(\theta) = \exp\{-(\theta' Q \theta)^{\alpha/2} + i\theta'\mu\}$, where Q is a positive definite matrix called the dispersion matrix. By comparing this characteristic function to the one in equation (16), we can see that the distribution of R belongs to the elliptical class. The dispersion matrix is defined by

$$Q = [\frac{R_{i,j}}{2}],\ where\ \frac{R_{i,j}}{2} = [R_i; R_j]_\alpha \, \|R_j\|_\alpha^{2-\alpha} \tag{22}$$

The *covariation* between two symmetric stable Paretian random variables with the same α is defined by

$$[R_i; R_j]_\alpha = \int_{S_2} s_i s_j^{<\alpha-1>} \Gamma(ds)$$

where $x^{<k>} = |x|^k sign(x)$. It can be shown that when $\alpha = 2$, $[R_1; R_2]_\alpha = \frac{1}{2}cov(R_{1,}R_2)$. The *variation* is defined as $[R_i; R_i]_\alpha = \|R_j\|_\alpha^\alpha$.

Since Z is elliptically distributed, by the results in the last section, the linear combination of its components $w'\not{R}\tilde{}E_1(w'\mu, Q, \phi)$ for some characteristic generator ϕ. Then the scale parameter $\sigma_{w'\not{R}}$, which is just the tracking error under this particular case, should be

$$\sigma_{w'R} = \sqrt{w'Qw}$$

where Q is determined by (22). The derivative of the tracking error is then similar to the one under the Gaussian case derived in section (5.1.1).

5.2 Quantile Derivatives(VaR)

Though VaR is shown to be a problematic risk measure, we still want to exploit its differentiability because the derivative of the expected shortfall depends on the derivative of the quantile measure. Under the assumption of the Gaussian or the more general elliptical distributions, it is not hard to calculate the derivatives of VaR as shown above, which can also be obtained by rescaling the variance-covariance matrix because the VaR is a linear function of the tracking error under the elliptical distributions ([12]). We present here the general case. We assume that there exists a well-defined joint density function for the random vector R.[14]

Proposition 5.2 *Let R be an $N \times 1$ random vector with the joint probability density function of $f(x_{1,} \ldots, x_n)$ satisfying $P[X = VaR_\alpha(X)] \neq 0$ and $\rho(X) = \rho(m'R) = VaR_\alpha(X) = VaR_\alpha$ be the risk measure defined in (7). Then,*

$$\frac{\partial VaR_\alpha(X)}{\partial m_i} = -E[R_i| - X = VaR_\alpha(X)],\ i = 1, \ldots, n \tag{23}$$

Proof. First consider a bivariate case of a random vector (Y, Z) with a smooth $p.d.f.\ f(y, z)$.Define VaR_α by

$$P[Y + m'R \geq VaR_\alpha] = \alpha$$

[14]Tasche (2000) discusses a slightly more general assumption, where he only assumes the existence of the conditional density function of X_i given $(X_{-i}) := (X_j)_{j\neq i}$. He notes that the existence of joint density implies the existence of the conditional counterpart but not necessarily vice versa.

Let $m \neq 0$,

$$\int \int_{(VaR_\alpha - m'R)} f(y,z)dydz = \alpha$$

Taking the derivative with respect to m, we have

$$\int (\frac{\partial VaR_\alpha}{\partial m} - z) f(y = VaR_\alpha - mz, z)dz = 0$$

Then

$$\frac{\partial VaR_\alpha}{\partial m} \int f(VaR_\alpha - mz, z)dz = \int zf(VaR_\alpha - mz, z)dz$$

Since $\int f(VaR_\alpha - mz, z)dz = P[Y + mZ = VaR_\alpha] \neq 0$, we have

$$\frac{\partial VaR_\alpha}{\partial m} = \frac{\int zf(VaR_\alpha - mz, z)dz}{\int f(VaR_\alpha - mz, z)dz} = E[Z|Y + mZ = VaR_\alpha]$$

Now replace $Y = -\sum_{j \neq i}^{n} m_j R_j$ and $m = m_i$ and $Z = -R_i,$ we have for all i,

$$\frac{\partial VaR_\alpha}{\partial m_i} = \frac{\partial \rho(X)}{\partial m_i} = -E[R_i| - m'R = VaR_\alpha]$$

■

The risk contribution, defined as $m_i \frac{\partial \rho(X)}{\partial m_i}$ in (4), is $-m_i E[R_i| - m'R = VaR_\alpha]$ in the case of VaR.

5.3 Differentiating The Expected Shortfall

The expected shortfall can be written in terms of an interval of VaR.(cf. [2]) This representation facilitates differentiating the expected shortfall because we already know the derivative of VaR.

Proposition 5.3 *(Tasche 2002) Let X be a random variable, $q_\alpha(X)$ the α-quantile defined in (7) for $\alpha \in (0,1)$ and $f : \mathbb{R} \to [0,\infty)$ a function such that $E|f(X)|^- < \infty$. Then ,*

$$\int_0^a f(q_u(X))du = E[f(X)1_{\{X \leq q_\alpha(X)\}}] + f(q_\alpha(X))(\alpha - P[X \leq q_\alpha(X)]) \quad (24)$$

Proof. Consider a uniformly distributed random variable U on $[0,1]$. We claim that the random variable $Z = q_U(X)$ has the same distribution as X, because $P(Z \le x) = P(q_U(X) \le x) = P(F_X^{-}(U) \le x) = P(U \le F_X(x)) = F_X(x)$. Since $u \to q_U(X)$ is non-decreasing we have

$$\begin{aligned} \{U &\le \alpha\} \subset \{q_U(X) \le q_\alpha(X)\} && (25) \\ \{U &> \alpha\} \cap \{q_U(X) \le q_\alpha(X)\} \subset \{q_U(X) = q_\alpha(X)\} && (26) \end{aligned}$$

(25) implies that $\{U > \alpha\} \cap \{q_U(X) \le q_\alpha(X)\} + \{U \le \alpha\} = \{q_U(X) \le q_\alpha(X)\}$.Then

$$\begin{aligned} \int_0^a f(q_u(X))du &= E_U[f(Z)1_{\{U\le\alpha\}}] \\ &= E_U[f(Z)1_{\{Z\le q_\alpha(X)\}}] - E_U[f(Z)1_{\{U>\alpha\}\cap\{Z\le q_\alpha(X)\}}] \\ &= E[f(X)1_{\{X\le q_\alpha(X)\}}] + f(q_\alpha(X))P[\{U>\alpha\}\cap\{X\le q_\alpha(X)\}] \\ &= E[f(X)1_{\{X\le q_\alpha(X)\}}] + f(q_\alpha(X))(\alpha - P[X \le q_a(X)) \end{aligned}$$

■

Corollary 5.4 *Given the VaR and the expected shortfall defined in (7) and (8), the following is true:*

$$ES_\alpha = \frac{1}{\alpha}\int_0^\alpha VaR_u(X)du \tag{27}$$

Proof. Let $f(X) = X$. By (8) and (24), we have

$$\begin{aligned} \int_0^a f(q_u(X))du &= \int_0^a q_u(X)du \\ &= E[f(X)1_{\{X\le q_\alpha(X)\}}] + q_\alpha(X)(\alpha - P[X \le q_\alpha(X)]) = -\alpha ES_\alpha \end{aligned}$$

Then dividing both sides by $-\alpha$ and replacing $-q_u(X)$ by $VaR_u(X)$ yield the result. ■

Proposition 5.5 *The partial derivative of ES_α defined in (8) is*

$$\frac{\partial ES_\alpha}{\partial m_i} = -\frac{1}{a}\{E[R_i 1_{\{X\le q_\alpha(X)\}}] + E[R_i|X = q_u(X)](\alpha - P[X \le q_\alpha(X)])\} \tag{28}$$

Proof. Given the representation in (27) and by (23), we have

$$\begin{aligned}\frac{\partial ES_\alpha}{\partial m_i} &= \frac{1}{\alpha}\int_0^\alpha \frac{\partial VaR_u}{\partial m_i}du \qquad (29)\\ &= -\frac{1}{\alpha}\int_0^\alpha E[R_i|-X = VaR_u(X)]du\\ &= -\frac{1}{\alpha}\int_0^\alpha E[R_i|X = q_u(X)]du \qquad (30)\end{aligned}$$

We can apply Proposition 5.3 again. Let $f(x) = E[R_i|X = x]$, then

$$\int_0^\alpha E[R_i|X = q_u(X)]du = E[E[R_i|X]1_{\{X\leq q_\alpha(X)\}}]+E[R_i|X = q_u(X)](\alpha-P[X \leq q_\alpha(X)]) \qquad (31)$$

The first term

$$\begin{aligned}E[E[R_i|X]1_{\{X\leq q_\alpha(X)\}}] &= E\{E[R_i|X]|X \leq q_\alpha(X)\} \cdot P[X \leq q_\alpha(X)]\\ &= E[R_i|X \leq q_\alpha(X)\} \cdot P[X \leq q_\alpha(X)]\\ &= E[R_i 1_{\{X\leq q_\alpha(X)\}}] \qquad (32)\end{aligned}$$

Then equation (31) becomes

$$\int_0^\alpha E[R_i|X = q_u(X)]du = E[R_i 1_{\{X\leq q_\alpha(X)\}}]+E[R_i|X = q_u(X)](\alpha-P[X \leq q_\alpha(X)])$$

Plugging this into (29) completes the proof. ■

6 Conclusion

We have reviewed the methodology of risk attribution, along with the properties of different risk measures and their calculation of derivatives. The rationale of risk attribution is that risk managers need to know where the major source of risk in their portfolio come from. The stand-alone risk statistics are useless in identifying the source of risk because of the presence of correlations with other assets. The partial derivative of the risk measure is justified to be an appropriate measure of risk contributions and therefore can help locating the major risk.

Having a good measure of risk is critical for risk attribution. A good risk measure should at least satisfy the coherent criterion. The widely accepted

measure of volatility could be a poor measure when the distribution is not symmetric. VaR is doomed to be a history because of its non-subadditivity and non-convexity. The conditional VaR or expected shortfall seems promising and is expected to become a dominant risk measure widely adopted in risk management.

Yet there are still some questions that haven't been answered. The statistical methods of estimating the risk contribution terms need to be further studied. Under the more general assumption of the distribution, the risk attribution results might be more accurate. The limitation of risk attribution analysis is that the risk contribution figure is a marginal concept. Risk attribution serves the purpose of hedging the major risk, which is closely related to portfolio optimization. How exactly the information extracted from the risk attribution process can be used in the portfolio optimization process still needs to be exploited. The two processes seem to be interdependent. Their interactions and relationship could be the topic of further studies.

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Appendix

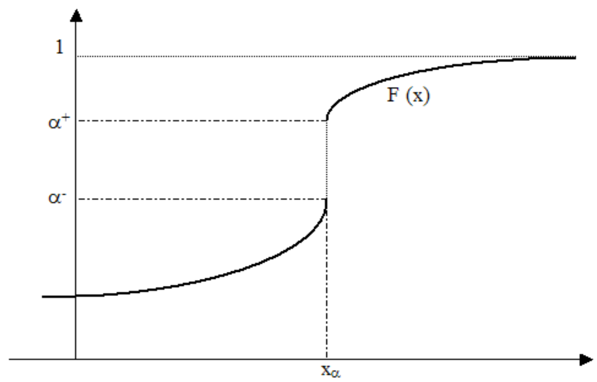

Figure 1: A jump occurs at $x_\alpha = VaR_\alpha$ of the distribution function $F(x)$ and there are more than one confidence levels (α^-, α^+) which give the same VaR.

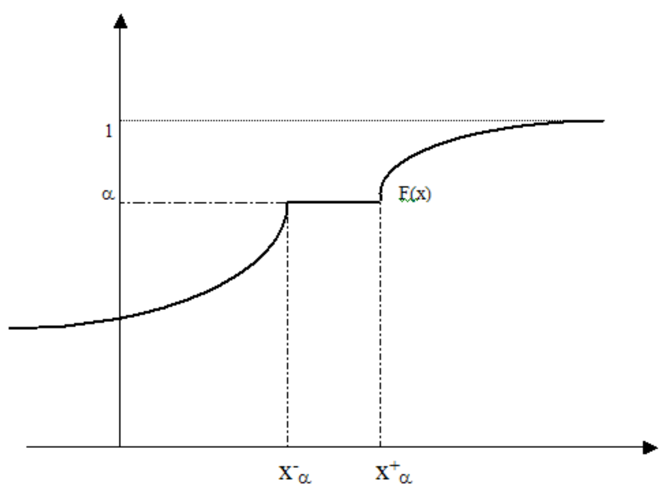

Figure 2: There are more than one candidate (x_α^-, x_α^+) of the VaR for the same confidence level.

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The Degree of Convergence of Over-iterated Positive Linear Operators

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Dedicated to Acad. Prof. Dr. Dr. h. c. D. D. Stancu

Abstract. The present note proposes a method to determine the degree of approximation for the iterates of certain positive linear operators towards B_1, the first Bernstein operator. Most of these operators are derived from a class of summation-type operators. We apply this method also to some continuous type operators: the Beta operators of the second kind.

Keywords: Positive linear operators, iterates, variation-diminishing splines, degree of approximation, contraction principle, second order modulus, second moments.

Classification: 41A15, 41A25, 41A36, 47H10.

1 Introduction

The present paper is motivated by a recent result of O. Agratini & I. Rus [1] (see also [28]) who proved convergence for over-iteration of certain general discretely defined operators. Over-iteration means that for a fixed operator its m-th powers are investigated when m goes to infinity. Their proof is very elegant and uses the contraction principle. In order to recall Agratini's and Rus' argument, we prove a generalization of their theorem first, and for a whole class of summation-type operators.

The operators are defined by $L_n : C[0,1] \to C[0,1]$ with

$$L_n(f;x) := \sum_{k=0}^{n} \psi_{n,k}(x) \cdot a_{n,k}(f), \tag{1}$$

where $\psi_{n,k}(x) \geq 0$, $a_{n,k}$ are linear positive functionals with $a_{n,k}e_0 = 1$, $k = 0, \ldots, n$, and $a_{n,0}(f) = f(0)$, $a_{n,n}(f) = f(1)$, $f \in C[0,1]$. With the supplementary condition that these operators reproduce linear functions we have the following relations:

$$\sum_{k=0}^{n} \psi_{n,k}(x) = 1, \text{ and } \sum_{k=0}^{n} \psi_{n,k}(x) \cdot a_{n,k}(e_1) = x, \ x \in [0,1].$$

Via the *contraction principle* (see, e.g., [2], [3]) we prove the following result.

Theorem 1.1 *Let L_n, $n \in \mathbb{N}$ fixed, be the operators given above. Define $u_n := \min_{x\in[0,1]}(\psi_{n,0}(x) + \psi_{n,n}(x))$. If $u_n > 0$, then the iterates $(L_n^m f)_{m\geq 1}$ with $f \in C[0,1]$ converge uniformly toward the linear function that interpolates f at the endpoints 0 and 1, i.e.,*

$$\lim_{m\to\infty} L_n^m(f;x) = f(0) + (f(1) - f(0))x, \ f \in C[0,1].$$

Proof. Consider the Banach space $(C[0,1], \|\cdot\|_\infty)$ where $\|\cdot\|_\infty$ is the Chebyshev norm. Let

$$X_{\alpha,\beta} := \{f \in C[0,1] : f(0) = \alpha, \ f(1) = \beta\}, \quad \alpha, \beta \in \mathbb{R}.$$

We note that

a) $X_{\alpha,\beta}$ is a closed subset of $C[0,1]$;

b) $C[0,1] = \bigcup\limits_{\alpha,\beta\in\mathbb{R}} X_{\alpha,\beta}$ is a partition of $C[0,1]$;

c) $X_{\alpha,\beta}$ is an invariant subset of L_n for all $\alpha,\beta \in \mathbb{R},\ n \in \mathbb{N}$, since the reproduction of linear functions implies interpolation of the function at the endpoints, i.e., $L_n(f;0) = f(0)$ and $L_n(f;1) = f(1)$.

Now we show that

$$L_n\,|_{X_{\alpha,\beta}} \colon X_{\alpha,\beta} \to X_{\alpha,\beta}$$

is a contraction for all $\alpha,\beta \in \mathbb{R}$.

Let $f,g \in X_{\alpha,\beta}$. We can write

$$\begin{aligned} |L_n(f;x) - L_n(g;x)| &= \left|\sum_{k=1}^{n-1} \psi_{n,k}(x)\cdot a_{n,k}(f-g)\right| \\ &\le \sum_{k=1}^{n-1} \psi_{n,k}(x)\cdot \|a_{n,k}\|\cdot\|f-g\|_\infty \\ &= (1-\psi_{n,0}(x)-\psi_{n,n}(x))\cdot\|f-g\|_\infty \\ &\le (1-u_n)\cdot\|f-g\|_\infty. \end{aligned} \tag{2}$$

Hence $\|L_nf - L_ng\|_\infty \le (1-u_n)\cdot\|f-g\|_\infty$ with $u_n > 0$ and thus $L_n\,|_{X_{\alpha,\beta}}$ is contractive.

On the other hand $\alpha + (\beta-\alpha)\cdot e_1 \in X_{\alpha,\beta}$ is a fixed point for L_n. If $f \in C[0,1]$ is arbitrarily given, then $f \in X_{f(0),f(1)}$ and from the contraction principle we have

$$\lim_{m\to\infty} L_n^m f = f(0) + (f(1)-f(0))e_1.\ \square$$

Note that the above proof is restricted to a *fixed* operator L_n and its iterates L_n^m. Furthermore, the proof goes only through for operators having a contraction constant $(1-u_n) < 1$. However, there are cases in which we do not have $u_n > 0$, but still convergence of the iterates takes place.

Both disadvantages are the main motivation of the present note. In Section 2 we prove general inequalities for the iterates of positive linear operators which are given in the

spirit of the paper by S. Karlin and Z. Ziegler [16] and were obtained for classical Bernstein operators in a slightly weaker form first in [11].

In Section 3 we give a full quantitative version of the result of O. Agratini and I. Rus [1]. In the subsequent two sections we consider several examples. Among them are Lupaş' Beta operators of the second kind and Schoenberg spline operators with equidistant knots. In both cases the approach of the aforementioned authors is not applicable, but our quantitative assertions show that over-iteration still leads to analogous results.

2 A general quantitative result

In the sequel we will consider again $L_n : C[0,1] \to C[0,1]$. However, relaxing the assumption of Section 1 we will consider general positive linear operators which reproduce linear functions. Note that in this section there will be no contraction argument.

The following estimate holds.

Theorem 2.1 *If L_n is given as above, for $m, n \in \mathbb{N}$ we have*

$$|L_n^m(f;x) - B_1(f;x)| \leq \frac{9}{4} \cdot \omega_2\left(f; \sqrt{L_n^m(e_1 \cdot (e_0 - e_1); x)}\right), \tag{3}$$

where $f \in C[0,1]$, $x \in [0,1]$, B_1 is the first Bernstein operator, and $e_i(t) = t^i$, $i \geq 0$.

Proof. For $g \in C^2[0,1]$ arbitrarily chosen we have the following estimate

$$\begin{aligned} |L_n^m(f;x) - B_1(f;x)| &\leq |(L_n^m - B_1)(f-g;x)| + |(L_n^m - B_1)(g;x)| \\ &\leq (\|L_n^m\|_\infty + \|B_1\|_\infty) \cdot \|f-g\|_\infty + |(L_n^m - B_1)(g;x)| \\ &\leq 2 \cdot \|f-g\|_\infty + |(L_n^m - B_1)(g;x)|. \end{aligned}$$

Since both of the operators L_n^m and B_1 reproduce linear functions, we have

$$L_n^m(B_1 g) = B_1(B_1 g) \in \prod\nolimits_1,$$

the polynomials of degree ≤ 1. Now we can evaluate

$$\begin{aligned} |(L_n^m - B_1)(g;x)| &= |L_n^m(g;x) - B_1(g;x) - L_n^m(B_1 g;x) + B_1(B_1 g;x)| \\ &= |L_n^m(g - B_1 g;x)| \leq L_n^m(|g - B_1 g|;x) \\ &\leq L_n^m\left(\frac{1}{2}\cdot\|g''\|_\infty \cdot e_1(e_0 - e_1);x\right) \\ &= \frac{1}{2}\cdot\|g''\|_\infty \cdot L_n^m(e_1(e_0 - e_1);x). \end{aligned}$$

Thus

$$|L_n^m(f;x) - B_1(f;x)| \leq 2\cdot\|f - g\|_\infty + \frac{1}{2}\|g''\|_\infty \cdot L_n^m(e_1(e_0 - e_1);x).$$

We substitute now $g := Z_h f$, Zhuk's function [36], which satisfies:

$$\|f - Z_h f\| \leq \frac{3}{4}\cdot\omega_2(f;h),\ \ \|(Z_h f)''\| \leq \frac{3}{2}\cdot h^{-2}\cdot\omega_2(f;h),\ 0 < h \leq \frac{1}{2}.$$

Zhuk's functions were also discussed in [13].

Hence we get

$$|L_n^m(f;x) - B_1(f;x)| \leq 2\cdot\frac{3}{4}\cdot\omega_2(f;h) + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{h^2}\ cdot L_n^m(e_1(e_0 - e_1);x)\cdot\omega_2(f;h),$$

with $h > 0$. Taking $h := \sqrt{L_n^m(e_1(e_0 - e_1);x)}$ yields the desired result. $\square$

Lemma 2.2 *Under the same assumptions on the operator L_n as above, we have*

$$0 \leq L_n(e_1(e_0 - e_1);x) \leq x(1-x)\left[1 - \min_{x\in(0,1)}\frac{L_n((e_1 - x)^2;x)}{x(1-x)}\right]. \tag{4}$$

Proof. Due to the linearity of the operator L_n and the fact that it preserves linear functions, one can easily observe that $L_n(e_1(e_0 - e_1);x) = x(1-x) - L_n((e_1 - x)^2;x)$. Thus,

$$\begin{aligned} 0 \leq L_n(e_1(e_0 - e_1);x) &= x(1-x)\left[1 - \frac{L_n((e_1 - x)^2;x)}{x(1-x)}\right],\ x\in(0,1), \\ &\leq x(1-x)\left[1 - \min_{x\in(0,1)}\frac{L_n((e_1 - x)^2;x)}{x(1-x)}\right]. \ \square \end{aligned}$$

For our further discussion we will exclude those operators whose second moments have zeros in the interior of the interval, $[0,1]$ in our case.

Theorem 2.3 *Let $L_n : C[0,1] \to C[0,1]$ be positive linear operators which preserve linear functions. We also suppose that there exists $\varepsilon_n > 0$ such that*

$$\varepsilon_n \cdot x(1-x) \le L_n((e_1 - x)^2; x), \ x \in [0,1]. \tag{5}$$

Then we have

$$0 \le L_n^m(e_1(e_0 - e_1); x) \le x(1-x) \cdot (1-\varepsilon_n)^m, \ m \in \mathbb{N}. \tag{6}$$

Proof. We will prove the above statement by induction. First we take $m = 1$. The imposed condition (5) can be rewritten as $\varepsilon_n \le \frac{L_n((e_1-x)^2;x)}{x(1-x)}$ for $x \in (0,1)$ implying

$$\varepsilon_n \le \min_{x\in(0,1)} \frac{L_n((e_1 - x)^2; x)}{x(1-x)}.$$

Thus inequality (4) yields

$$L_n(e_1(e_0 - e_1); x) \le x(1-x)(1-\varepsilon_n).$$

We assume the relation

$$L_n^m(e_1(e_0 - e_1); x) \le x(1-x)(1-\varepsilon_n)^m$$

to be true for a fixed $m \in \mathbb{N}$ and shall prove it for $m+1$. Indeed, we have

$$L_n^{m+1}(e_1(e_0 - e_1); x) \le (1-\varepsilon_n)^m \cdot L_n(e_1(e_0 - e_1); x) \le x(1-x) \cdot (1-\varepsilon_n)^{m+1}.$$

Hence it follows that the estimate (6) is true for all $m \in \mathbb{N}$ $\square$.

In case that $\varepsilon_n < 1$ (which occurs often), by combining the above theorem and Theorem 2.1 we get the following

Corollary 2.4 *With the same assumptions on the operator L_n as above and (5) we get*

$$|L_n^m(f; x) - B_1(f; x)| \le \frac{9}{4} \cdot \omega_2\left(f; \sqrt{x(1-x)(1-\varepsilon_n)^m}\right), f \in C[0,1], \ x \in [0,1]. \tag{7}$$

Note that the operator L_n now is *not* necessarily fixed. We can thus – as was done by Karlin and Ziegler – consider $\lim_{n\to\infty} L_n^{m_n}$ were m_n depends on n such that $\lim_{n\to\infty} (1-\varepsilon_n)^{m_n} \to 0$ and still get uniform convergence towards $B_1 f$.

3 Discretely defined operators

In this section we show that the general result from Section 2 implies the convergence assertion of Agratini and Rus, also providing a full quantitative version of it. Our assertion is given in terms of the second order modulus, the best to be expected under the present conditions. However, due to the use of the contraction constant $(1-u_n)$ some pointwise information is lost.

We return to the operators considered in Section 1. For a given partition on $[0,1]$ such that $0=x_{n,0}<x_{n,1}<\ldots<x_{n,n}=1$ we specialize the functionals $a_{n,k}$ by assuming

$$a_{n,k}(f)=f(x_{n,k}),\ k=0,\ldots,n.$$

We obtain

$$L_n(f;x)=\sum_{k=0}^{n}\psi_{n,k}(x)\cdot f(x_{n,k}),\ f\in C[0,1],\ x\in[0,1]. \tag{8}$$

Guided by a result of R.P. Kelisky & T.J. Rivlin [17], O. Agratini and I. Rus studied these operators L_n in [1]. It is known that operators L_n of this type have attracted attention for at least 100 years now. We mention here the interesting note of T. Popoviciu [25] who in turn refers to the classical book of É. Borel [7] , see also [8]. (Polynomial) operators of the given type also appear in H. Bohman's now classical paper [6] and in Butzer's problem (see, e.g., [9] and the references cited there for details). Further historical information can be found in A. Pinkus' most interesting work [22].

Lemma 3.1 *As in the first section we assume that the operators (8) reproduce linear functions. This implies that* $\psi_{n,0}(0)=\psi_{n,n}(1)=1$.

Proof. It is known that $L_ne_i=e_i,\ i=0,1,$ implies interpolation at the endpoints of the function, i.e., $L_n(f;0)=f(0)$ and $L_n(f;1)=f(1)$. This means that

$$f(0) \;=\; L_n(f;0)=\psi_{n,0}\cdot f(0)+\sum_{k=1}^{n}\psi_{n,k}(0)\cdot f(x_{n,k}) \text{ or}$$

$$(1-\psi_{n,0}(0))\cdot f(0) \;=\; \sum_{k=1}^{n}\psi_{n,k}(0)\cdot f(x_{n,k}),\ \text{for all } f\in C[0,1]. \tag{9}$$

We define $f \in C[0,1]$ by

$$f(x) := \begin{cases} -\frac{1}{x_{1,n}}\cdot x+1, & x\in[0,x_{1,n}] \\ 0, & x\in(x_{1,n},1]. \end{cases}$$

and substitute it into (9). Thus we easily arrive at $\psi_{n,0}(0) = 1$. In a similar way we can prove that $\psi_{n,n}(1) = 1$. □

Thus the conditions $\psi_{n,0}(0) = \psi_{n,n}(1) = 1$ are automatically satisfied. Furthermore, we will give pointwise and uniform estimates for these operators L_n which imply the result of O. Agratini and I. Rus.

First we have

Proposition 3.2 *For $L_n : C[0,1] \to C[0,1]$ defined as in (8) one has*

$$L_n^m(e_1(e_0-e_1);x) \le \frac{1}{4}\cdot(1-u_n)^{m-1}\cdot(1-\psi_{n,0}(x)-\psi_{n,n}(x)), \tag{10}$$

with $m \in \mathbb{N}$, $f \in C[0,1]$ and $x \in [0,1]$. Like in Theorem 1.1 the inequality $u_n = \min\limits_{x\in[0,1]}(\psi_{n,0}(x)+\psi_{n,n}(x)) > 0$ is assumed.

Proof. We will prove this statement by induction. For $m = 1$ we have

$$\begin{aligned} L_n(e_1(e_0-e_1);x) &= L_n(e_1-e_2;x) = \sum_{k=0}^{n}(x_{n,k}-x_{n,k}^2)\cdot\psi_{n,k}(x) \\ &= \sum_{k=1}^{n-1}(x_{n,k}-x_{n,k}^2)\cdot\psi_{n,k}(x) \le \frac{1}{4}\cdot\sum_{k=1}^{n-1}\psi_{n,k}(x) \\ &= \frac{1}{4}\cdot(1-\psi_{n,0}(x)-\psi_{n,n}(x)). \end{aligned}$$

We suppose now that the relation

$$L_n^m(e_1(e_0-e_1);x) \le \frac{1}{4}\cdot(1-u_n)^{m-1}\cdot(1-\psi_{n,0}(x)-\psi_{n,n}(x))$$

is true for a fixed $m \in \mathbb{N}$. We show it for $m+1$. We apply on this relation the operator L_n, obtaining

$$L_n^{m+1}(e_1(e_0-e_1);x) \;\le\; \frac{1}{4}(1-u_n)^{m-1}L_n(1-\psi_{n,0}-\psi_{n,n};x)$$

$$\begin{aligned}
&= \frac{1}{4}(1-u_n)^{m-1} L_n\left(\sum_{k=1}^{n-1}\psi_{n,k};x\right)\\
&= \frac{1}{4}(1-u_n)^{m-1} \sum_{l=0}^{n}\sum_{k=1}^{n-1}\psi_{n,k}(x_{n,l})\psi_{n,l}(x)\\
&= \frac{1}{4}(1-u_n)^{m-1} \sum_{l=1}^{n-1}\sum_{k=1}^{n-1}\psi_{n,k}(x_{n,l})\psi_{n,l}(x)\\
&= \frac{1}{4}(1-u_n)^{m-1} \sum_{l=1}^{n-1}\psi_{n,l}(x)\cdot\sum_{k=1}^{n-1}\psi_{n,k}(x_{n,l})\\
&\le \frac{1}{4}(1-u_n)^{m} \sum_{l=1}^{n-1}\psi_{n,l}(x) = \frac{1}{4}(1-u_n)^m\cdot(1-\psi_{n,0}(x)-\psi_{n,n}(x)).
\end{aligned}$$

We have thus proved that the relation (10) is true for any $m \in \mathbb{N}$. □

Remark 3.3 Uniformly one has

$$L_n^m(e_1(e_0-e_1)) \le \frac{1}{4}\cdot(1-u_n)^m. \tag{11}$$

The following pointwise estimate is a consequence of Theorem 2.1.

Proposition 3.4 *For the considered operators L_n we have*

$$|L_n^m(f;x)-B_1(f;x)| \le \frac{9}{4}\cdot\omega_2\left(f;\frac{1}{2}\cdot\sqrt{(1-u_n)^{m-1}\cdot(1-\psi_{n,0}(x)-\psi_{n,n}(x))}\right).$$

The latter inequality reflects the fact that the iterates interpolate B_1f (and f) at $x=0$ and $x=1$.

Corollary 3.5 *The uniform estimate is also easily obtained from Theorem 2.1 and (11) as*

$$\|L_n^m f - B_1 f\|_\infty \le \frac{9}{4}\cdot\omega_2\left(f;\frac{1}{2}\cdot\sqrt{(1-u_n)^m}\right).$$

Note that the contraction constant $1-u_n<1$ figures repeatedly in the above inequalities.

4 Applications I

We have two essentially different quantitative assertions by now, one from Section 2 and one from Section 3. In this section we consider a group of operators to which both results are applicable.

4.1 Bernstein-Sheffer-Popoviciu operators

In this subsection we shall investigate the approximation order of the iterates of some binomial type operators. They were studied among others by T. Popoviciu [23] and P. Sablonnière [29].

We recall their definition, using the classical notations.

Let $Q \in \Sigma_\delta$ be a delta operator, $(p_n)_{n\geq 0}$ its basic sequence and the known associated power series $\varphi(t) = c_1 + c_2 t + \ldots$. The operators $L_n^Q : C[0,1] \to C[0,1]$, defined by

$$L_n^Q(f;x) := \frac{1}{p_n(1)} \cdot \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(1-x) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}, \tag{12}$$

with $p_n(1) \neq 0$ are called *Bernstein-Sheffer-Popoviciu* operators.

Remark 4.1 One can easly observe that these operators can be derived from (8) by considering the equidistant knot sequence $x_{k,n} = \frac{k}{n},\ k = 0, \ldots, n$, and making a proper substitution of the fundamental functions $\psi_{n,k}$.

Some of their properties we collect in

Property 4.2 *Let L_n^Q be the considered binomial type operators.*

(i) L_n^Q *are linear and positive, under some conditions regarding the coefficients of the associated power series, i.e.,* $c_1 > 0$ *and* $c_n \geq 0,\ n \geq 2$.

(ii) L_n^Q *reproduce linear functions.*

(iii) $L_n^Q((e_1-x)^2;x) = \left(1-\frac{n-1}{n}\cdot\frac{(Q'^{-2}p_{n-2})(1)}{p_n(1)}\right)\cdot x(1-x),\ n\geq 2,$ *where* Q' *is the Pincherle derivative of the delta operator* Q*. It is known that there always exists* Q'^{-1}*, for* Q *a delta operator.*

(iv) Uniform convergence of the operator towards the identity operator is implied iff $\lim\limits_{n\to\infty}\frac{(Q'^{-2}p_{n-2})(1)}{p_n(1)}=1.$

For more details about the theory of binomial type operators see G. C. Rota [27] and also the monography [32]. It implies that also this operator integrates perfectly in our approximation scheme, because $\varepsilon_n = 1-\frac{n-1}{n}\cdot\frac{(Q'^{-2}p_{n-2})(1)}{p_n(1)} < 1$ for sufficiently large n, due to (iv). We consider this to be true for $n \geq N$.

Applying Corollary 2.4 we obtain

Proposition 4.3 *For the iterates of the* L_n^Q *we can state*

$$|L_n^{Qm}(f;x) - B_1(f;x)| \leq \frac{9}{4}\cdot\omega_2\left(f;\sqrt{x(1-x)\left(\frac{n-1}{n}\cdot\frac{Q'^{-2}(p_{n-2},1)}{p_n(1)}\right)^m}\right),$$

for $n\geq N,\ f\in C[0,1],\ x\in[0,1].$

4.2 Bernstein - Stancu operators

Setting the delta operator $Q := \Delta_{-\alpha},\ \alpha\geq 0$ and its corresponding basic sequence $p_n(x) = x^{[n,-\alpha]}$ we obtain the operators considered by D. D. Stancu [31] in 1968. Thus the operators $S_n^{(\alpha)} : C[0,1]\to\prod_n$ were defined by

$$S_n^{(\alpha)}(f;x) := \sum_{k=0}^{n} w_{n,k}^{(\alpha)}(x)\cdot f\left(\frac{k}{n}\right),\ f\in C[0,1],\ x\in[0,1],\ \alpha\geq 0,$$

where

$$w_{n,k}^{(\alpha)}(x) := \binom{n}{k}\frac{x^{[k,-\alpha]}\cdot(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}},\ k=0,\ldots,n.$$

Here $y^{[m,-\alpha]}$ is a factorial power with the step $-\alpha$, i.e.,

$$\begin{aligned} y^{[0,-\alpha]} &= 1,\\ y^{[m,-\alpha]} &= y\cdot(y+\alpha)\ldots(y+(m-1)\alpha),\ m\in\mathbb{N}. \end{aligned}$$

The following facts were proved in the latter paper.

Property 4.4 *(i)* $S_n^{(\alpha)}$ *are linear and positive.*

(ii) $S_n^{(\alpha)}$ *reproduce linear functions.*

(iii) $S_n^{(\alpha)}((e_1 - x)^2; x) = x(1-x) \cdot \frac{1}{n} \cdot \frac{1+n\alpha}{1+\alpha}$.

This implies that the operators defined by D. D. Stancu satisfy the requirements of our theorem and we can consider $\varepsilon_n = \frac{1}{n} \cdot \frac{1+n\alpha}{1+\alpha} < 1$.

Taking into account Corollary 2.4 we arrive at

Proposition 4.5 *Let* $S_n^{(\alpha)},\ n \in \mathbb{N},\ \alpha \geq 0$ *be a sequence of Stancu operators. For* $m \in \mathbb{N},\ f \in C[0,1]$ *and* $x \in [0,1]$ *we have*

$$|S_n^{(\alpha)^m}(f;x) - B_1(f;x)| \leq \frac{9}{4} \cdot \omega_2 \left(f; \sqrt{x(1-x) \left(1 - \frac{1}{n} \cdot \frac{1+n\alpha}{1+\alpha} \right)^m} \right).$$

4.3 The classical Bernstein operators

For $\alpha = 0$ we arrive at the classical Bernstein operators the second moments of which were used by T. Popoviciu as early as 1942 (see [24], cf. also [26]). An early paper on over-iterated Bernstein operators is - besides the one by R.P. Kelisky & T.J. Rivlin - an article of P.C. Sikkema [30]. Using Proposition 4.5 immediately yields

Proposition 4.6 *Let* $B_n,\ n \in \mathbb{N}$, *be the sequence of Bernstein operators. For* $m \in \mathbb{N},\ f \in C[0,1]$ *and* $x \in [0,1]$ *we obtain*

$$|B_n^m(f;x) - B_1(f;x)| \leq \frac{9}{4} \cdot \omega_2 \left(f, \sqrt{x(1-x) \left(1 - \frac{1}{n} \right)^m} \right).$$

A similar result was first obtained by H. Gonska in [11] with a constant 4 instead of $\frac{9}{4}$, and as a special consequence of a more general quantitative result for the approximation

of finitely defined operators (see [20] and [12] for further details). More information on iterated Bernstein operators can be found in the recent note [14].

Remark 4.7

(i) I. Gavrea and D. H. Mache [10] discussed a certain special case of the general operators (1). Restricting ourselves to a special situation, their operators were defined by

$$A_n(f;x) := \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} \cdot a_{n,k}(f). \tag{13}$$

Here $a_{n,k} : C[0,1] \to \mathbb{R}$ are positive linear functionals verifying $a_{n,k}e_0 = 1$ and $a_{n,k}e_1 = \frac{k}{n}$, $k = 0, \ldots, n$ (the latter condition being our special situation). Hence linear functions are reproduced so that Theorem 2.1 is applicable. We also note that $A_n(f;0) = f(0)$ and $A_n(f;1) = f(1)$, which is true for every positive linear operator reproducing linear functions. This implies that $a_{n,0}(f) = f(0)$ and $a_{n,n}(f) = f(1)$. The special form of the fundamental functions implies that we can take $u_n = \frac{1}{2^{n-1}}$ to arrive - in a way analogous to Proposition 3.4 - at

$$|A_n^m(f;x) - B_1(f;x)| \le \frac{9}{4} \cdot \omega_2 \left(f; \frac{1}{2} \cdot \sqrt{\left(1 - \frac{1}{2^{n-1}}\right)^{m-1} (1-(1-x)^n - x^n)} \right).$$

The genuine Bernstein - Durrmeyer operator (see, e.g., [15], [21], [34]) has this particular form.

(ii) A further class of positive linear operators which generalize the Bernstein operators was recently introduced by N. Vornicescu [33]. His operators use general knots

$$0 = x_0 < x_1 < \ldots < x_n = 1$$

and reproduce linear functions so that the general results from Theorem 2.1 and Proposition 3.4 are also applicable.

5 Applications II

Here we consider two types of operators to which the approach of Agratini and Rus is not applicable. The Beta operators in the next subsection are not discretely defined,

and the Schoenberg spline operators are such that $u_n = 0$ so that the contraction argument fails in this case.

5.1 Beta operators of the second kind

Here we discuss an example which is not covered by the ansatz of Section 1. In his German Ph. D. thesis [18] A. Lupaş defined the sequence of operators $\bar{\mathbb{B}}_n : C[0,1] \to B[0,1]$, given by

$$\bar{\mathbb{B}}_n(f;x) := \begin{cases} f(0), & x = 0, \\ \frac{1}{B(nx, x-nx)} \cdot \int\limits_0^1 t^{nx-1}(1-t)^{n-1-nx} \cdot f(t)dt, & 0 < x < 1, \\ f(1), & x = 1. \end{cases} \tag{14}$$

We will call them *Beta operators of the second kind* (since earlier in his thesis A. Lupaş considered Beta operators of the first kind).

The author proved the following.

Property 5.1 *(i)* $\bar{\mathbb{B}}_n$ *are linear and positive,*

(ii) $\bar{\mathbb{B}}_n$ *reproduce linear functions,*

(iii) $\bar{\mathbb{B}}_n((e_1 - x)^2; x) = \frac{x(1-x)}{n+1}$.

In this case $\varepsilon_n = \frac{1}{n+1} < 1,\ n \geq 1$. We get the following pointwise estimate.

Proposition 5.2 *Let* $\bar{\mathbb{B}}_n$ *be a sequence of Beta operators of the second kind. Let* $m \in \mathbb{N},\ f \in C[0,1]$ *and* $x \in [0,1]$. *Then we have*

$$|\bar{\mathbb{B}}_n^m(f;x) - B_1(f;x)| \leq \frac{9}{4} \cdot \omega_2\left(f; \sqrt{x(1-x)\left(1 - \frac{1}{n+1}\right)^m}\right).$$

5.2 Schoenberg spline operators on equidistant knots

The contraction principle, very efficient in many cases, is not applicable in the case of Schoenberg splines, since one cannot find a contraction constant strictly less than 1. One motivation for this paper was to propose a method that yields relevant results also for the iterates of Schoenberg splines. So far, we succeded for certain cases with equidistant knots.

Consider the knot sequence $\Delta_n = \{x_i\}_{-k}^{n+k}$, $2 \le k \le n-1$ with

$$\Delta_n : x_{-k} = \ldots = x_0 = 0 < x_1 < x_2 < \ldots < x_n = \ldots x_{n+k} = 1,$$

and $x_i = \frac{i}{n}$ for $0 \le i \le n$.

For a function $f \in \mathbb{R}^{[0,1]}$, the variation- diminishing spline of degree k w.r.t. Δ_n is given by

$$\begin{aligned} S_{n,k}(f;x) &= \sum_{j=-k}^{n-1} f(\xi_{j,k}) \cdot N_{j,k}(x),\ 0 \le x < 1, \\ S_{n,k}(f;1) &= \lim_{y \to 1, y<1} S_{n,k}(f;y). \end{aligned}$$

$\xi_{j,k}$ are the Greville abscissas

$$\xi_{j,k} := \frac{x_{j+1} + \ldots + x_{j+k}}{k},\ -k \le j \le j \le n-1,$$

and the fundamental functions are the normalized B-splines

$$N_{j,k}(x) := (x_{j+k+1} - x_j)[x_j, x_{j+1}, \ldots, x_{j+k+1}](\cdot - x)_+^k.$$

The following proposition will provide a possible choice for ε_n.

Proposition 5.3 *For the second moments of the latter operators we have the lower estimate*

$$\min\left\{\frac{2}{21n^2}, \frac{1}{21n(k-1)}\right\} \cdot x(1-x) \le S_{n,k}((e_1 - x)^2; x),\ 2 \le k \le n-1.$$

Proof. The following lower bound of the second moments was given in [5] (see also [4]). For $2 \le k \le n-1$ one has

$$\frac{S_{n,k}((e_1-x)^2;x)}{x(1-x)} \ge c_k \cdot \frac{\min\left\{2x(1-x), \frac{k}{n}\right\}}{n(k-1)x(1-x)} \ge c_k \cdot \frac{\min\left\{2, \frac{k}{n} \cdot \frac{1}{x(1-x)}\right\}}{n(k-1)},$$

where $c_k = \frac{9}{88} \ge \frac{1}{10}$ for $k \ge 3$ and $c_2 = \frac{3}{124} \ge \frac{1}{42}$.

We consider now two cases:

First case. For $2k > n$ and $2 \le k \le n-1$ we have $\min\left\{2, \frac{k}{n} \cdot \frac{1}{x(1-x)}\right\} = 2$. Thus,

$$\frac{S_{n,k}((e_1-x)^2;x)}{x(1-x)} \ge \frac{1}{21n(k-1)} \text{ for } n < 2k.$$

Second case. If $n \ge 2k$, then $\min\left\{2, \frac{k}{n} \cdot \frac{1}{x(1-x)}\right\} \ge \frac{4k}{n}$. We have $\frac{S_{n,k}((e_1-x)^2;x)}{x(1-x)} \ge c_k \cdot \frac{4}{n^2} \cdot \frac{k}{k-1}$. This estimate can be carried out further, since $\frac{k}{k-1} \ge 1$ and $c_k \ge \frac{1}{42}$. We arrive at

$$\frac{S_{n,k}((e_1-x)^2;x)}{x(1-x)} \ge \frac{2}{21n^2} \text{ for } n \ge 2k. \ \Box$$

Remark 5.4 The above proposition delivers us one possible value of

$$\varepsilon_{n,k} = \min\left\{\frac{2}{21n^2}, \frac{1}{21n(k-1)}\right\} < 1,$$

with $2 \le k \le n-1$. One can observe that for $k=1$ the condition (5) is not verified, because the second moment of the piecewise linear operator has zeros in the interior of the interval (e.g., see A. Lupaş [19]). It is also clear that $S_{n,1}^m f = S_{n,1} f,\ m \ge 1$.

Now we can easily derive a convergence result for the iterates of the Schoenberg spline operator.

Proposition 5.5 *For $S_{n,k}$, $2 \le k \le n-1$, defined as above we have*

$$|S_{n,k}^m(f;x) - B_1(f;x)| \le \frac{9}{4}\omega_2\left(f; \sqrt{x(1-x)\left(1-\min\left\{\frac{2}{21n^2}, \frac{1}{21n(k-1)}\right\}\right)^m}\right).$$

Remark 5.6 For $2 \leq k \leq n-1$ fixed we have $\lim_{m\to\infty}(1-\varepsilon_{n,k})^m = 0$. Thus $\lim_{m\to\infty} S_{n,k}^m f = B_1 f$. An analogous convergence result also holds for more general knot sequences, as shown by H. J. Wenz in [35]. Due to the lack of a suitable lower bound for the second moments we have not been able yet to give quantitative results in this general case.

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The Lidstone interpolation on tetrahedron*

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Abstract

We give a Lidstone interpolation formula, including its remainder, for a real function defined on a tetrahedron. These results extend the corresponding ones given for triangle by F.A. Costabile and F. Dell'Accio in a recent paper.

We also present some numerical examples.

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Key words and phrases: Lidstone interpolation on triangle and tetrahedron, remainder term, Peano Theorem.

1 Preliminaries

The Lidstone interpolation was introduced in 1920 and is a two-point interpolation process utilizing even derivatives [7]. As it is mentioned in [5], there exist few results in the literature regarding the extension of approximation of univariate functions by means of Lidstone polynomials to functions of two variables over non rectangular domains.

In this paper we extend some univariate approximation formulas to functions of three variables defined on a tetrahedron. The extension to the bidimensional case has been considered in [5].

We recall first some classical results regarding Lidstone interpolation [1], [2].

The Lidstone polynomial is the unique polynomial Λ_n of degree $2n+1$, $n \in \mathbb{N}$, defined on the interval $[0,1]$ by

$$\begin{aligned}
&\Lambda_0(x) = x, \\
&\Lambda_n''(x) = \Lambda_{n-1}(x), \\
&\Lambda_n(0) = \Lambda_n(1) = 0, \quad n \geq 1.
\end{aligned} \tag{1}$$

For a given function f possessing a sufficient number of derivatives, the Lidstone interpolation problem consists in finding a polynomial of degree $2n-1$ satisfying the Lidstone interpolation conditions,

$$\begin{aligned}
&(L_n f)^{(2k)}(0) = f^{(2k)}(0), \\
&(L_n f)^{(2k)}(1) = f^{(2k)}(1), \quad 0 \leq k \leq n-1
\end{aligned} \tag{2}$$

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According to [1], the Lidstone interpolant $L_n f$ uniquely exists and can be expressed as

$$(L_n f)(x) = \sum_{k=0}^{n-1} \left[f^{(2k)}(0)\Lambda_k(1-x) + f^{(2k)}(1)\Lambda_k(x) \right]. \tag{3}$$

Remark 1 *The Lidstone operator L_n is exact for the polynomials of degree not greater than $2n-1$, $n \in \mathbb{N}$.*

The Lidstone interpolation formula is

$$f = L_n f + R_n f, \tag{4}$$

where $R_n f$ denotes the remainder.

For $f \in C^{2n}[0,1]$ one can apply Peano's Theorem, [1], [2], and obtain

$$(R_n f)(x) = \int_0^1 g_n(x,s) f^{(2n)}(s) ds, \tag{5}$$

with

$$g_n(x,s) = \begin{cases} -\sum\limits_{k=0}^{n-1} \Lambda_k(x) \frac{(1-s)^{2n-2k-1}}{(2n-2k-1)!}, & x \le s \\ -\sum\limits_{k=0}^{n-1} \Lambda_k(1-x) \frac{s^{2n-2k-1}}{(2n-2k-1)!}, & s \le x. \end{cases}$$

2 The Lidstone interpolation formula on tetrahedron

We consider the tetrahedron

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 \,|\, x,y,z \ge 0,\ x+y+z \le 1 \right\},$$

and let f be a real function from $C^{2n}(T)$, $n \in \mathbb{N}$.

We consider the Lidstone interpolation on a segment from $\mathbb{R}^3$. Similarly to [5, Lemma 3.1], where the segment was in $\mathbb{R}^2$, we obtain the following result:

Lemma 2 *Let D be a convex domain in $\mathbb{R}^3$, $f \in C^{2n}(D)$, $n \in \mathbb{N}$, and consider two points $P(a,b,c), Q(u,v,w) \in D$. Let $l(\lambda) = (x(\lambda), y(\lambda), z(\lambda))$, $\lambda \in [0,1]$, $l(0) = P$, $l(1) = Q$ be a linear parametrization of the segment PQ and consider $h \equiv x'(\lambda)$, $k \equiv y'(\lambda)$ and $p \equiv z'(\lambda)$, $\lambda \in [0,1]$. We have*

$$f(x(\lambda), y(\lambda), z(\lambda)) = (L_n f)(x(\lambda), y(\lambda), z(\lambda)) + (R_n f)(x(\lambda), y(\lambda), z(\lambda)),$$

with

$$\begin{aligned} &(L_n f)(x(\lambda), y(\lambda), z(\lambda)) = \qquad (6) \\ &= \sum_{i=0}^{n-1} \sum_{\alpha_1+\alpha_2+\alpha_3=2i} \tfrac{(2i)!}{\alpha_1!\alpha_2!\alpha_3!} h^{\alpha_1} k^{\alpha_2} p^{\alpha_3} \Big[\tfrac{\partial^{2i}}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} f(a,b,c) \Lambda_i(1-\lambda) \\ &\qquad + \tfrac{\partial^{2i}}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} f(u,v,w) \Lambda_i(\lambda) \Big] \end{aligned}$$

and

$$(R_nf)(x(\lambda),y(\lambda),z(\lambda)) = \tag{7}$$
$$= \sum_{\alpha_1+\alpha_2+\alpha_3=2n} \int_0^1 \tfrac{(2n)!}{\alpha_1!\alpha_2!\alpha_3!} h^{\alpha_1}k^{\alpha_2}p^{\alpha_3} \tfrac{\partial^{2n}}{\partial x^{\alpha_1}\partial y^{\alpha_2}\partial z^{\alpha_3}} f(x(s),y(s),z(s))g_n(\lambda,s)ds.$$

Proof. Let us denote $\varphi(\lambda)=(f\circ l)(\lambda)$. We apply the interpolation formula (4) for the function φ:

$$\varphi(\lambda)=(L_n\varphi)(\lambda)+(R_n\varphi)(\lambda),$$

with

$$(L_n\varphi)(\lambda)=\sum_{i=0}^{n-1}\left[\varphi^{(2i)}(0)\Lambda_i(1-\lambda)+\varphi^{(2i)}(1)\Lambda_i(\lambda)\right] \tag{8}$$

and by (5)

$$(R_n\varphi)(\lambda)=\int_0^1 g_n(\lambda,s)\varphi^{(2n)}(s)ds. \tag{9}$$

We have

$$\varphi^{(k)}(\lambda)=\sum_{\alpha_1+\alpha_2+\alpha_3=k} \tfrac{k!}{\alpha_1!\alpha_2!\alpha_3!} h^{\alpha_1}k^{\alpha_2}p^{\alpha_3} \tfrac{\partial^{k}}{\partial x^{\alpha_1}\partial y^{\alpha_2}\partial z^{\alpha_3}}(f\circ l)(\lambda), \tag{10}$$

for $1\le k\le 2n$. Replacing (10) in (8) and (9), we obtain (6) and (7), respectively. ■

We shall restrict further to the cases when the segment PQ is in one of the three coordinate planes, i.e., when $c=w=0$ or $b=v=0$ or $a=u=0$. Therefore, we consider the derivatives of f in the directions of $\nu_1\big(\frac{h}{\sqrt{h^2+k^2}},\frac{k}{\sqrt{h^2+k^2}},0\big)$, $\nu_2\big(\frac{h}{\sqrt{h^2+p^2}},0,\frac{p}{\sqrt{h^2+p^2}}\big)$ and respectively $\nu_3\big(0,\frac{k}{\sqrt{k^2+p^2}},\frac{p}{\sqrt{k^2+p^2}}\big)$. The polynomial given by (6) satisfies the following interpolation conditions corresponding to the previous three cases:

$$\begin{aligned}
&\tfrac{d^{2k}}{d\lambda^{2k}}(L_nf)(x(\lambda),y(\lambda),0)\big|_{\lambda=0}=\tfrac{\partial^{2k}}{\partial\nu_1^{2k}}f(a,b,0),\\
&\tfrac{d^{2k}}{d\lambda^{2k}}(L_nf)(x(\lambda),y(\lambda),0)\big|_{\lambda=1}=\tfrac{\partial^{2k}}{\partial\nu_1^{2k}}f(u,v,0),\\
&\tfrac{d^{2k}}{d\lambda^{2k}}(L_nf)(x(\lambda),0,z(\lambda))\big|_{\lambda=0}=\tfrac{\partial^{2k}}{\partial\nu_2^{2k}}f(a,0,c),\\
&\tfrac{d^{2k}}{d\lambda^{2k}}(L_nf)(x(\lambda),0,z(\lambda))\big|_{\lambda=1}=\tfrac{\partial^{2k}}{\partial\nu_2^{2k}}f(u,0,w),\\
&\tfrac{d^{2k}}{d\lambda^{2k}}(L_nf)(0,y(\lambda),z(\lambda))\big|_{\lambda=0}=\tfrac{\partial^{2k}}{\partial\nu_3^{2k}}f(0,b,c),\\
&\tfrac{d^{2k}}{d\lambda^{2k}}(L_nf)(0,y(\lambda),z(\lambda))\big|_{\lambda=1}=\tfrac{\partial^{2k}}{\partial\nu_3^{2k}}f(0,v,w),\quad k=0,...,n-1,
\end{aligned}\tag{11}$$

with the remainders of the interpolation formulas given by:

$$\begin{aligned}
&(R_nf)(x(\lambda),y(\lambda),0)=(h^2+k^2)^n\int_0^1 \tfrac{\partial^{2n}}{\partial\nu_1^{2n}}f(x(\lambda),y(\lambda),0)g_n(\lambda,s)ds,\\
&(R_nf)(x(\lambda),0,z(\lambda))=(h^2+p^2)^n\int_0^1 \tfrac{\partial^{2n}}{\partial\nu_2^{2n}}f(x(\lambda),0,z(\lambda))g_n(\lambda,s)ds,\\
&(R_nf)(0,y(\lambda),z(\lambda))=(k^2+p^2)^n\int_0^1 \tfrac{\partial^{2n}}{\partial\nu_3^{2n}}f(0,y(\lambda),z(\lambda))g_n(\lambda,s)ds.
\end{aligned}\tag{12}$$

These results follow from chain rule (or deriving (6) and the conditions (2)) and taking into account the definitions of the derivatives in a direction,

$$\frac{\partial^{2n}}{\partial \nu_1^{2n}} f(x(\lambda), y(\lambda), 0) = \frac{1}{(h^2+k^2)^n} \sum_{j=0}^{2n} \binom{2n}{j} h^{2n-j} k^j \frac{\partial^{2n}}{\partial x^{2n-j}\partial y^j} f(x(\lambda), y(\lambda), 0),$$

$$\frac{\partial^{2n}}{\partial \nu_2^{2n}} f(x(\lambda), 0, z(\lambda)) = \frac{1}{(h^2+p^2)^n} \sum_{j=0}^{2n} \binom{2n}{j} h^{2n-j} p^j \frac{\partial^{2n}}{\partial x^{2n-j}\partial z^j} f(x(\lambda), y(\lambda), 0),$$

$$\frac{\partial^{2n}}{\partial \nu_3^{2n}} f(0, y(\lambda), z(\lambda)) = \frac{1}{(k^2+p^2)^n} \sum_{j=0}^{2n} \binom{2n}{j} k^{2n-j} p^j \frac{\partial^{2n}}{\partial x^{2n-j}\partial z^j} f(0, y(\lambda), z(\lambda)).$$

We give now the main result of this paper.

Theorem 3 *Let $n \in \mathbb{N}$, $\nu = (-1, 1, 0)$ and f be a real function from $C^{4n-2}(T)$. The Lidstone interpolation conditions*

$$\frac{\partial^{2i}}{\partial x^{2i-j}\partial y^j} (L_n^T f)(0,0,0) = \frac{\partial^{2i}}{\partial x^{2i-j}\partial y^j} f(0,0,0), \quad i = 0, ..., n-1,\ j = 0, ..., 2i \tag{13}$$

and

$$\begin{aligned}
&\frac{\partial^{2i}}{\partial z^{2i-j}\partial \nu^j} (L_n^T f)(0,0,1) = \frac{\partial^{2i}}{\partial z^{2i-j}\partial \nu^j} f(0,0,1), \\
&\frac{\partial^{2i}}{\partial x^{2i-j}\partial \nu^j} (L_n^T f)(1,0,0) = \frac{\partial^{2i}}{\partial x^{2i-j}\partial \nu^j} f(1,0,0), \\
&\frac{\partial^{2i}}{\partial y^{2i-j}\partial \nu^j} (L_n^T f)(0,1,0) = \frac{\partial^{2i}}{\partial y^{2i-j}\partial \nu^j} f(0,1,0), \quad i = 0, ..., n-1,\ j = 0, 2, ..., 2i
\end{aligned} \tag{14}$$

are satisfied by the following rational function

$$\begin{aligned}
&(L_n^T f)(x,y,z) = \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-i-1} \sum_{l=0}^{n-i-k-1} \binom{2i}{j} (-1)^j \left(\tfrac{x+y}{x+y+z}\right)^{2i} \cdot \\
&\cdot \Bigg[\Lambda_k\left(1 - \tfrac{x+y}{x+y+z}\right) \Lambda_i\left(\tfrac{x}{x+y}\right) \Lambda_l(1-x-y-z) f^{(2i-j+2k,j,2l)}(0,0,0) \\
&\quad + \Lambda_k\left(1 - \tfrac{x+y}{x+y+z}\right) \Lambda_i\left(\tfrac{x}{x+y}\right) \Lambda_l(x+y+z) f^{(2i-j+2k,j,2l)}(0,0,1) \\
&\quad + \Lambda_k\left(\tfrac{x+y}{x+y+z}\right) \Lambda_i\left(\tfrac{x}{x+y}\right) \Lambda_l(1-x-y-z) f^{(2i-j+2l,j+2k,0)}(0,0,0) \\
&\quad + \Lambda_k\left(\tfrac{x+y}{x+y+z}\right) \Lambda_i\left(\tfrac{x}{x+y}\right) \Lambda_l(x+y+z) f^{(2i-j+2l,j+2k,0)}(1,0,0) \\
&\quad + \Lambda_k\left(1 - \tfrac{x+y}{x+y+z}\right) \Lambda_i\left(\tfrac{y}{x+y}\right) \Lambda_l(1-x-y-z) f^{(2i-j,j+2k,2l)}(0,0,0) \\
&\quad + \Lambda_k\left(1 - \tfrac{x+y}{x+y+z}\right) \Lambda_i\left(\tfrac{y}{x+y}\right) \Lambda_l(x+y+z) f^{(2i-j,j+2k,2l)}(0,0,1) \\
&\quad + \Lambda_k\left(\tfrac{x+y}{x+y+z}\right) \Lambda_i\left(\tfrac{y}{x+y}\right) \Lambda_l(1-x-y-z) f^{(2i-j,j+2k+2l,0)}(0,0,0) \\
&\quad + \Lambda_k\left(\tfrac{x+y}{x+y+z}\right) \Lambda_i\left(\tfrac{y}{x+y}\right) \Lambda_l(x+y+z) f^{(2i-j,j+2k+2l,0)}(0,1,0) \Bigg].
\end{aligned} \tag{15}$$

The remainder of the Lidstone interpolation formula on tetrahedron,

$$f(x,y,z) = (L_n^T f)(x,y,z) + (R_n^T f)(x,y,z), \tag{16}$$

is given by

$$(R_n^T f)(x,y,z)= \tag{17}$$
$$=\sum_{i=0}^{n-1}\sum_{k=0}^{n-i-1} 2^i\big(\tfrac{x+y}{x+y+z}\big)^{2i}\cdot$$
$$\cdot\Big[\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\int_0^1 \tfrac{\partial^{2n+2i+2k}}{\partial\nu^{2i}\partial x^{2k}\partial z^{2n}} f(0,0,s)g_n(x+y+z,s)ds$$
$$+\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\int_0^1 \tfrac{\partial^{2n+2i+2k}}{\partial\nu^{2i}\partial x^{2n}\partial y^{2k}} f(s,0,0)g_n(x+y+z,s)ds$$
$$+\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{y}{x+y}\big)\int_0^1 \tfrac{\partial^{2n+2i+2k}}{\partial\nu^{2i}\partial y^{2k}\partial z^{2n}} f(0,0,s)g_n(x+y+z,s)ds$$
$$+\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{y}{x+y}\big)\int_0^1 \tfrac{\partial^{2n+2i+2k}}{\partial\nu^{2i}\partial y^{2n+2k}} f(0,s,0)g_n(x+y+z,s)ds\Big]$$
$$+\sum_{i=0}^{n-1} 2^i\big(\tfrac{x+y}{x+y+z}\big)^{2i}\cdot$$
$$\cdot\Big[\Lambda_i\big(\tfrac{x}{x+y}\big)\int_0^1 \tfrac{\partial^{2n}}{\partial x^{2(n-i)}\partial\nu^{2i}} f((x+y+z)s,0,(x+y+z)(1-s))g_{n-i}\big(\tfrac{x+y}{x+y+z},s\big)ds$$
$$+\Lambda_i\big(\tfrac{y}{x+y}\big)\int_0^1 \tfrac{\partial^{2n}}{\partial y^{2(n-i)}\partial\nu^{2i}} f(0,(x+y+z)s,(x+y+z)(1-s))g_{n-i}\big(\tfrac{x+y}{x+y+z},s\big)ds\Big]$$
$$+2^n(x+y)^{2n}\int_0^1 \tfrac{\partial^{2n}}{\partial\nu^{2n}} f((1-s)(x+y),s(x+y),z)g_n\big(\tfrac{y}{x+y},s\big)ds.$$

Proof. First, we consider Lidstone interpolation on the triangle OMN, $O(0,0,0)$, $M(1,0,0)$, $N(0,1,0)$ according to [5]. The corresponding interpolation formula is

$$f=L_n^{\Delta OMN}f+R_n^{\Delta OMN}f, \tag{18}$$

with

$$(L_n^{\Delta OMN}f)(x,y,0)=$$
$$=\sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1}\tbinom{2i}{j}(-1)^j(x+y)^{2i}\Lambda_k(1-x-y)\Lambda_i\big(\tfrac{x}{x+y}\big)f^{(2i-j+2k,j,0)}(0,0,0)$$
$$+\sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1}\tbinom{2i}{j}(-1)^j(x+y)^{2i}\Lambda_k(x+y)\Lambda_i\big(\tfrac{x}{x+y}\big)f^{(2i-j,j+2k,0)}(1,0,0)$$
$$+\sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1}\tbinom{2i}{j}(-1)^j(x+y)^{2i}\Lambda_k(1-x-y)\Lambda_i\big(\tfrac{y}{x+y}\big)f^{(2i-j,j+2k,0)}(0,0,0)$$
$$+\sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1}\tbinom{2i}{j}(-1)^j(x+y)^{2i}\Lambda_k(x+y)\Lambda_i\big(\tfrac{y}{x+y}\big)f^{(2i-j,j+2k,0)}(0,1,0).$$

and $R_n^{\Delta OMN} f$ is the remainder and it is given by:

$$(R_n^{\Delta OMN} f)(x,y,z) =$$
$$= \sum_{i=0}^{n-1} 2^i (x+y)^{2i} \Big[\Lambda_i\big(\tfrac{x}{x+y}\big) \int_0^1 \tfrac{\partial^{2n}}{\partial x^{2(n-i)} \partial \nu^{2i}} f(s,0,0) g_{n-i}(x+y,s) ds$$
$$+ \Lambda_i\big(\tfrac{y}{x+y}\big) \int_0^1 \tfrac{\partial^{2n}}{\partial y^{2(n-i)} \partial \nu^{2i}} f(0,s,0) g_{n-i}(x+y,s) ds \Big]$$
$$+ 2^n (x+y)^{2n} \int_0^1 \tfrac{\partial^{2n}}{\partial \nu^{2n}} f((1-s)(x+y), s(x+y), 0) g_n\big(\tfrac{y}{x+y}, s\big) ds.$$

By means of the affine transformation $h : \mathbb{R}^3 \to \mathbb{R}$,

$$h(x,y,z) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ -a & -a & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix},$$

with $a = x+y+z$, we can obtain from (18) the interpolation formula on the triangle ABC, $A(x+y+z,0,0)$, $B(0,x+y+z,0)$, $C(0,0,x+y+z)$ with $x+y+z = \alpha$, $\alpha \in (0,1)$:

$$f = L_n^{\Delta ABC} f + R_n^{\Delta ABC} f,$$

$$(L_n^{\Delta ABC} f)(x,y,z) =$$
$$= \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-i-1} \tbinom{2i}{j} (-1)^j \big(\tfrac{x+y}{x+y+z}\big)^{2i} \Lambda_k\big(1 - \tfrac{x+y}{x+y+z}\big) \Lambda_i\big(\tfrac{x}{x+y}\big) f^{(2i-j+2k,j,0)}(0,0,x+y+z)$$
$$+ \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-i-1} \tbinom{2i}{j} (-1)^j \big(\tfrac{x+y}{x+y+z}\big)^{2i} \Lambda_k\big(\tfrac{x+y}{x+y+z}\big) \Lambda_i\big(\tfrac{x}{x+y}\big) f^{(2i-j,j+2k,0)}(x+y+z,0,0)$$
$$+ \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-i-1} \tbinom{2i}{j} (-1)^j \big(\tfrac{x+y}{x+y+z}\big)^{2i} \Lambda_k\big(1 - \tfrac{x+y}{x+y+z}\big) \Lambda_i\big(\tfrac{y}{x+y}\big) f^{(2i-j,j+2k,0)}(0,0,x+y+z)$$
$$+ \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-i-1} \tbinom{2i}{j} (-1)^j \big(\tfrac{x+y}{x+y+z}\big)^{2i} \Lambda_k\big(\tfrac{x+y}{x+y+z}\big) \Lambda_i\big(\tfrac{y}{x+y}\big) f^{(2i-j,j+2k,0)}(0,x+y+z,0).$$

and

$$(R_n^{\Delta ABC} f)(x,y,z) = \tag{19}$$
$$= \sum_{i=0}^{n-1} 2^i \big(\tfrac{x+y}{x+y+z}\big)^{2i} \cdot$$
$$\cdot \Big[\Lambda_i\big(\tfrac{x}{x+y}\big) \int_0^1 \tfrac{\partial^{2n}}{\partial x^{2(n-i)} \partial \nu^{2i}} f((x+y+z)s, 0, (x+y+z)(1-s)) g_{n-i}\big(\tfrac{x+y}{x+y+z}, s\big) ds$$
$$+ \Lambda_i\big(\tfrac{y}{x+y}\big) \cdot \int_0^1 \tfrac{\partial^{2n}}{\partial y^{2(n-i)} \partial \nu^{2i}} f(0, (x+y+z)s, (x+y+z)(1-s)) g_{n-i}\big(\tfrac{x+y}{x+y+z}, s\big) ds \Big]$$
$$+ 2^n (x+y)^{2n} \int_0^1 \tfrac{\partial^{2n}}{\partial \nu^{2n}} f((1-s)(x+y), s(x+y), z) g_n\big(\tfrac{y}{x+y}, s\big) ds.$$

By making the notation

$$A_{ij}(x,y,z) = \tbinom{2i}{j} (-1)^j \big(\tfrac{x+y}{x+y+z}\big)^{2i}, \tag{20}$$

we obtain that

$$
\begin{aligned}
&(L_n^{\Delta ABC} f)(x,y,z)= \qquad (21)\\
=&\sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1} A_{ij}(x,y,z)\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)f^{(2i-j+2k,j,0)}(0,0,x+y+z)\\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1} A_{ij}(x,y,z)\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)f^{(2i-j,j+2k,0)}(x+y+z,0,0)\\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1} A_{ij}(x,y,z)\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{y}{x+y}\big)f^{(2i-j,j+2k,0)}(0,0,x+y+z)\\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1} A_{ij}(x,y,z)\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{y}{x+y}\big)f^{(2i-j,j+2k,0)}(0,x+y+z,0).
\end{aligned}
$$

By Lemma 2 we obtain the interpolant on the segment with endpoints $(0,0,0)$ and $(1,0,0)$:

$$
\begin{aligned}
(L_n^{[1]} f)(x,0,0)&=\sum_{i=0}^{n-1}\left(\tfrac{\partial^{2i}}{\partial x^{2i}}f(0,0,0)\Lambda_i(1-x)+\tfrac{\partial^{2i}}{\partial x^{2i}}f(1,0,0)\Lambda_i(x)\right),\\
(L_n^{[1]} f)(x+y+z,0,0)&=\sum_{i=0}^{n-1}\Big(\tfrac{\partial^{2i}}{\partial x^{2i}}f(0,0,0)\Lambda_i(1-x-y-z)\\
&\qquad+\tfrac{\partial^{2i}}{\partial x^{2i}}f(1,0,0)\Lambda_i(x+y+z)\Big).
\end{aligned}
$$

We have the interpolation formula

$$
f(x+y+z,0,0)=(L_n^{[1]} f)(x+y+z,0,0)+(R_n^{[1]} f)(x+y+z,0,0),
$$

where the remainder $R_n^{[1]}(f)(x+y+z,0,0)$ is obtained from (12):

$$
(R_n^{[1]} f)(x+y+z,0,0)=\int_0^1 \tfrac{\partial^{2n}}{\partial x^{2n}}f(s,0,0)g_n(x+y+z,s)ds.
$$

Hence,

$$
\begin{aligned}
&f^{(2i-j,j+2k,0)}(x+y+z,0,0)= \qquad (22)\\
=&L_n^{[1]}\left(\tfrac{\partial^{2i+2k}}{\partial x^{2i-j}\partial y^{j+2k}}f(x+y+z,0,0)\right)+R_n^{[1]}\left(\tfrac{\partial^{2i+2k}}{\partial x^{2i-j}\partial y^{j+2k}}f\right)(x+y+z,0,0)\\
=&\sum_{l=0}^{n-i-k-1}\Big(f^{(2i-j+2l,j+2k,0)}(0,0,0)\Lambda_l(1-x-y-z)+f^{(2i-j+2l,j+2k,0)}(1,0,0)\Lambda_l(x+y+z)\Big)\\
&+\int_0^1 \tfrac{\partial^{2n+2i+2k}}{\partial x^{2n+2i-j}\partial y^{j+2k}}f(s,0,0)g_n(x+y+z,s)ds.
\end{aligned}
$$

Again, by Lemma 2, we obtain the interpolant on the segment with endpoints $(0,0,0)$ and $(0,1,0)$:

$$
\begin{aligned}
(L_n^{[2]} f)(0,y,0)&=\sum_{i=0}^{n-1}\left(\tfrac{\partial^{2i}}{\partial y^{2i}}f(0,0,0)\Lambda_i(1-y)+\tfrac{\partial^{2i}}{\partial y^{2i}}f(0,1,0)\Lambda_i(y)\right),\\
(L_n^{[2]} f)(0,x+y+z,0)&=\sum_{i=0}^{n-1}\left(\tfrac{\partial^{2i}}{\partial y^{2i}}f(0,0,0)\Lambda_i(1-x-y-z)+\tfrac{\partial^{2i}}{\partial y^{2i}}f(0,1,0)\Lambda_i(x+y+z)\right).
\end{aligned}
$$

We have the interpolation formula

$$f(0,x+y+z,0)=(L_n^{[2]}f)(0,x+y+z,0)+(R_n^{[2]}f)(0,x+y+z,0),$$

where the remainder $(R_n^{[2]}f)(0,x+y+z,0)$ is obtained from (12):

$$(R_n^{[2]}f)(0,x+y+z,0)=\int_0^1 \tfrac{\partial^{2n}}{\partial y^{2n}}f(0,s,0)g_n(x+y+z,s)ds.$$

Hence,

$$\begin{aligned}
&f^{(2i-j,j+2k,0)}(0,x+y+z,0)= \qquad (23)\\
=&L_n^{[2]}\left(\tfrac{\partial^{2i+2k}}{\partial x^{2i-j}\partial y^{j+2k}}f(0,x+y+z,0)\right)+R_n^{[2]}\left(\tfrac{\partial^{2i+2k}}{\partial x^{2i-j}\partial y^{j+2k}}f\right)(0,x+y+z,0)\\
=&\sum_{l=0}^{n-i-k-1}\left(f^{(2i-j,j+2k+2l,0)}(0,0,0)\Lambda_l(1-x-y-z)+f^{(2i-j,j+2k+2l,0)}(0,1,0)\Lambda_l(x+y+z)\right)\\
&+\int_0^1 \tfrac{\partial^{2n+2i+2k}}{\partial x^{2i-j}\partial y^{2n+j+2k}}f(0,s,0)g_n(x+y+z,s)ds.
\end{aligned}$$

Finally, using once more Lemma 2, we obtain the interpolant on the segment with endpoints $(0,0,0)$ and $(0,0,1)$:

$$(L_n^{[3]}f)(0,0,z)=\sum_{i=0}^{n-1}\left(\tfrac{\partial^{2i}}{\partial y^{2i}}f(0,0,0)\Lambda_i(1-z)+\tfrac{\partial^{2i}}{\partial y^{2i}}f(0,0,1)\Lambda_i(z)\right)$$

$$(L_n^{[3]}f)(0,0,x+y+z)=\sum_{i=0}^{n-1}\left(\tfrac{\partial^{2i}}{\partial z^{2i}}f(0,0,0)\Lambda_i(1-x-y-z)+\tfrac{\partial^{2i}}{\partial z^{2i}}f(0,0,1)\Lambda_i(x+y+z)\right).$$

We have the interpolation formula

$$f(0,0,x+y+z)=(L_n^{[3]}f)(0,0,x+y+z)+(R_n^{[3]}f)(0,0,x+y+z),$$

where the remainder $(R_n^{[3]}f)(0,0,x+y+z)$ is obtained from (12):

$$(R_n^{[3]}f)(0,0,x+y+z)=\int_0^1 \tfrac{\partial^{2n}}{\partial z^{2n}}f(0,0,s)g_n(x+y+z,s)ds.$$

Hence,

$$\begin{aligned}
&f^{(2i-j,j+2k,0)}(0,0,x+y+z)= \qquad (24)\\
=&L_n^{[3]}\left(\tfrac{\partial^{2i+2k}}{\partial x^{2i-j}\partial y^{j+2k}}f\right)(0,0,x+y+z)+R_n^{[3]}\left(\tfrac{\partial^{2i+2k}}{\partial x^{2i-j}\partial y^{j+2k}}f\right)(0,0,x+y+z)\\
=&\sum_{l=0}^{n-i-k-1}\left(f^{(2i-j,j+2k,2l)}(0,0,0)\Lambda_l(1-x-y-z)+f^{(2i-j,j+2k,2l)}(0,0,1)\Lambda_l(x+y+z)\right)\\
&+\int_0^1 \tfrac{\partial^{2n+2i+2k}}{\partial x^{2i-j}\partial y^{j+2k}\partial z^{2n}}f(0,0,s)g_n(x+y+z,s)ds
\end{aligned}$$

and

$$\begin{aligned}
&f^{(2i-j+2k,j,0)}(0,0,x+y+z)= \qquad (25)\\
=&L_n^{[3]}\left(\tfrac{\partial^{2i+2k}}{\partial x^{2i-j+2k}\partial y^{j}}f\right)(0,0,x+y+z)+R_n^{[3]}\left(\tfrac{\partial^{2i+2k}}{\partial x^{2i-j+2k}\partial y^{j}}f\right)(0,0,x+y+z)\\
=&\sum_{l=0}^{n-i-k-1}\left(f^{(2i-j+2k,j,2l)}(0,0,0)\Lambda_l(1-x-y-z)+f^{(2i-j+2k,j,2l)}(0,0,1)\Lambda_l(x+y+z)\right)\\
&+\int_0^1 \tfrac{\partial^{2n+2i+2k}}{\partial x^{2i-j+2k}\partial y^{j}\partial z^{2n}}f(0,0,s)g_n(x+y+z,s)ds.
\end{aligned}$$

Next, replacing (22), (23), (24) and (25) in (21) we obtain

$$
\begin{aligned}
&(L_n^T f)(x,y,z) = \qquad (26)\\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} A_{ij}(x,y,z)\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\\
&\cdot\Big(f^{(2i-j+2k,j,2l)}(0,0,0)\Lambda_l(1-x-y-z)+f^{(2i-j+2k,j,2l)}(0,0,1)\Lambda_l(x+y+z)\Big)\\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} A_{ij}(x,y,z)\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\\
&\cdot\Big(f^{(2i-j+2l,j+2k,0)}(0,0,0)\Lambda_l(1-x-y-z)+f^{(2i-j+2l,j+2k,0)}(1,0,0)\Lambda_l(x+y+z)\Big)\\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} A_{ij}(x,y,z)\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{y}{x+y}\big)\\
&\cdot\Big(f^{(2i-j,j+2k,2l)}(0,0,0)\Lambda_l(1-x-y-z)+f^{(2i-j,j+2k,2l)}(0,0,1)\Lambda_l(x+y+z)\Big)\\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} A_{ij}(x,y,z)\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{y}{x+y}\big)\\
&\cdot\Big(f^{(2i-j,j+2k+2l,0)}(0,0,0)\Lambda_l(1-x-y-z)+f^{(2i-j,j+2k+2l,0)}(0,1,0)\Lambda_l(x+y+z)\Big),
\end{aligned}
$$

which, taking into account the notation for A_{ij}, results in (15).

Replacing (22), (23), (24) and (25) in (21) we also obtain the remainder term of the interpolation formula on tetrahedron:

$$
\begin{aligned}
&(R_n^T f)(x,y,z) = \qquad (27)\\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{2i}\sum_{k=0}^{n-i-1}\tbinom{2i}{j}(-1)^j\big(\tfrac{x+y}{x+y+z}\big)^{2i}\cdot\\
&\cdot\Big[\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\int_0^1 \tfrac{\partial^{2n+2i+2k}}{\partial x^{2i-j+2k}\partial y^j\partial z^{2n}} f(0,0,s)g_n(x+y+z,s)ds\\
&\quad+\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\int_0^1 \tfrac{\partial^{2n+2i+2k}}{\partial x^{2n+2i-j}\partial y^{j+2k}} f(s,0,0)g_n(x+y+z,s)ds\\
&\quad+\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{y}{x+y}\big)\int_0^1 \tfrac{\partial^{2n+2i+2k}}{\partial x^{2i-j}\partial y^{j+2k}\partial z^{2n}} f(0,0,s)g_n(x+y+z,s)ds\\
&\quad+\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{y}{x+y}\big)\int_0^1 \tfrac{\partial^{2n+2i+2k}}{\partial x^{2i-j}\partial y^{2n+j+2k}} f(0,s,0)g_n(x+y+z,s)ds\Big]\\
&+R_n^{\Delta ABC}f)(x,y,z).
\end{aligned}
$$

Applying in (27) the formula

$$
\sum_{j=0}^{2i}\tbinom{2i}{j}(-1)^j\frac{\partial^{2i}}{\partial x^{2i-j}\partial y^j}f = 2^i\frac{\partial^{2i}}{\partial \nu^{2i}}f \qquad (28)
$$

and replacing the expression of $R_n^{\Delta ABC}f(x,y,z)$ from (19) we obtain (17).

Next we prove that $L_n^T f$ is verifying the Lidstone interpolation condition given by (13) and (14). We have $\frac{\partial}{\partial \nu}=\frac{1}{\sqrt{2}}\big(-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\big)$. Taking into account that

$$
\sum_{j=0}^{2i}\tbinom{2i}{j}(-1)^{2i-j}\frac{\partial^{2i}}{\partial x^{2i-j}\partial y^j}f = 2^i\frac{\partial^{2i}}{\partial \nu^{2i}}f
$$

after we replace (20) in (15) we obtain the following expression

$$
\begin{aligned}
&(L_n^T f)(x,y,z) = \\
&= \sum_{i=0}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\big(\tfrac{x+y}{x+y+z}\big)^{2i}\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\Lambda_l(1-x-y-z)\tfrac{\partial^{2i+2k+2l}}{\partial x^{2k}\partial \nu^{2i}\partial z^{2l}}f(0,0,0)\\
&\quad+ \sum_{i=0}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\big(\tfrac{x+y}{x+y+z}\big)^{2i}\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\Lambda_l(x+y+z)\tfrac{\partial^{2i+2k+2l}}{\partial x^{2k}\partial \nu^{2i}\partial z^{2l}}f(0,0,1)\\
&\quad+ \sum_{i=0}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\big(\tfrac{x+y}{x+y+z}\big)^{2i}\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\Lambda_l(1-x-y-z)\tfrac{\partial^{2i+2k+2l}}{\partial x^{2l}\partial y^{2k}\partial \nu^{2i}}f(0,0,0)\\
&\quad+ \sum_{i=0}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\big(\tfrac{x+y}{x+y+z}\big)^{2i}\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\Lambda_l(x+y+z)\tfrac{\partial^{2i+2k+2l}}{\partial x^{2l}\partial y^{2k}\partial \nu^{2i}}f(1,0,0)\\
&\quad+ \sum_{i=0}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\big(\tfrac{x+y}{x+y+z}\big)^{2i}\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{y}{x+y}\big)\Lambda_l(1-x-y-z)\tfrac{\partial^{2i+2k+2l}}{\partial \nu^{2i}\partial y^{2k}\partial z^{2l}}f(0,0,0)\\
&\quad+ \sum_{i=0}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\big(\tfrac{x+y}{x+y+z}\big)^{2i}\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{y}{x+y}\big)\Lambda_l(x+y+z)\tfrac{\partial^{2i+2k+2l}}{\partial \nu^{2i}\partial y^{2k}\partial z^{2l}}f(0,0,1)\\
&\quad+ \sum_{i=0}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\big(\tfrac{x+y}{x+y+z}\big)^{2i}\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{y}{x+y}\big)\Lambda_l(1-x-y-z)\tfrac{\partial^{2i+2k+2l}}{\partial \nu^{2i}\partial y^{2k+2l}}f(0,0,0)\\
&\quad+ \sum_{i=0}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\big(\tfrac{x+y}{x+y+z}\big)^{2i}\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{y}{x+y}\big)\Lambda_l(x+y+z)\tfrac{\partial^{2i+2k+2l}}{\partial \nu^{2i}\partial y^{2k+2l}}f(0,1,0).
\end{aligned}
$$

It is not difficult to verify that

$$
\begin{aligned}
&\tfrac{\partial}{\partial\nu}\Big(\big(\tfrac{x+y}{x+y+z}\big)^{2i}\Lambda_l(1-x-y-z)\Big)=0,\\
&\tfrac{\partial}{\partial\nu}\Big(\big(\tfrac{x+y}{x+y+z}\big)^{2i}\Lambda_l(x+y+z)\Big)=0
\end{aligned}
$$

and

$$
\begin{aligned}
\tfrac{\partial}{\partial\nu}\Big(\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\Big) &= \Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\tfrac{\partial}{\partial\nu}\Lambda_i\big(\tfrac{x}{x+y}\big),\\
\tfrac{\partial}{\partial\nu}\Big(\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\Big) &= \Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\tfrac{\partial}{\partial\nu}\Lambda_i\big(\tfrac{x}{x+y}\big),
\end{aligned}
\tag{29}
$$

for $i=0,...,n-1$. We have

$$
\begin{aligned}
\tfrac{\partial}{\partial\nu}\Lambda_i\big(\tfrac{x}{x+y}\big) &= -\tfrac{1}{\sqrt{2}}\tfrac{1}{x+y}\Lambda_i'\big(\tfrac{x}{x+y}\big)\\
\tfrac{\partial^2}{\partial\nu^2}\Lambda_i\big(\tfrac{x}{x+y}\big) &= \tfrac{1}{2}\tfrac{1}{(x+y)^2}\Lambda_i''\big(\tfrac{x}{x+y}\big)
\end{aligned}
$$

We know from (1) that $\Lambda_n''(x)=\Lambda_{n-1}(x)$, thus

$$
\begin{aligned}
\tfrac{\partial^2}{\partial\nu^2}\Lambda_i\big(\tfrac{x}{x+y}\big) &= \tfrac{1}{2}\tfrac{1}{(x+y)^2}\Lambda_{i-1}\big(\tfrac{x}{x+y}\big)\\
&\vdots\\
\tfrac{\partial^{2m}}{\partial\nu^{2m}}\Lambda_i\big(\tfrac{x}{x+y}\big) &= \tfrac{1}{2^m}\tfrac{1}{(x+y)^{2m}}\Lambda_{i-m}\big(\tfrac{x}{x+y}\big)
\end{aligned}
\tag{30}
$$

and in analogous way we obtain

$$
\tfrac{\partial^{2m}}{\partial\nu^{2m}}\Lambda_i\big(\tfrac{y}{x+y}\big) = \tfrac{1}{2^m}\tfrac{1}{(x+y)^{2m}}\Lambda_{i-m}\big(\tfrac{y}{x+y}\big)
\tag{31}
$$

for each $m = 0, ..., n-1$, $i = 0, ..., n-1$, $k = 0, ..., n-i-1$. Taking into account (29), (30) and (31 we obtain

$$\frac{\partial^{2m}}{\partial\nu^{2m}}\Big(\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\Big) = \Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\tfrac{1}{2^m(x+y)^{2m}}\Lambda_{i-m}\big(\tfrac{x}{x+y}\big), \quad (32)$$
$$\frac{\partial^{2m}}{\partial\nu^{2m}}\Big(\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_i\big(\tfrac{x}{x+y}\big)\Big) = \Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\tfrac{1}{2^m(x+y)^{2m}}\Lambda_{i-m}\big(\tfrac{x}{x+y}\big).$$

Hence,

$$\begin{aligned}
&\frac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(x,y,z) = \\
&= \sum_{i=m}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^{i-m}\big(\tfrac{1}{x+y+z}\big)^{2i}(x+y)^{2i-2m}\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_{i-m}\big(\tfrac{x}{x+y}\big)\Lambda_l(1-x-y-z)\tfrac{\partial^{2i+2k+2l}}{\partial x^{2k}\partial\nu^{2i}\partial z^{2l}}f(0,0,0)\\
&+ \sum_{i=m}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^{i-m}\big(\tfrac{1}{x+y+z}\big)^{2i}(x+y)^{2i-2m}\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_{i-m}\big(\tfrac{x}{x+y}\big)\Lambda_l(x+y+z)\tfrac{\partial^{2i+2k+2l}}{\partial x^{2k}\partial\nu^{2i}\partial z^{2l}}f(0,0,1)\\
&+ \sum_{i=m}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^{i-m}\big(\tfrac{1}{x+y+z}\big)^{2i}(x+y)^{2i-2m}\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_{i-m}\big(\tfrac{x}{x+y}\big)\Lambda_l(1-x-y-z)f\tfrac{\partial^{2i+2k+2l}}{\partial x^{2l}\partial y^{2k}\partial\nu^{2i}}f(0,0,0)\\
&+ \sum_{i=m}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^{i-m}\big(\tfrac{1}{x+y+z}\big)^{2i}(x+y)^{2i-2m}\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_{i-m}\big(\tfrac{x}{x+y}\big)\Lambda_l(x+y+z)\tfrac{\partial^{2i+2k+2l}}{\partial x^{2l}\partial y^{2k}\partial\nu^{2i}}f(1,0,0)\\
&+ \sum_{i=m}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^{i-m}\big(\tfrac{1}{x+y+z}\big)^{2i}(x+y)^{2i-2m}\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_{i-m}\big(\tfrac{y}{x+y}\big)\Lambda_l(1-x-y-z)\tfrac{\partial^{2i+2k+2l}}{\partial\nu^{2i}\partial y^{2k}\partial z^{2l}}f(0,0,0)\\
&+ \sum_{i=m}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^{i-m}\big(\tfrac{1}{x+y+z}\big)^{2i}(x+y)^{2i-2m}\Lambda_k\big(1-\tfrac{x+y}{x+y+z}\big)\Lambda_{i-m}\big(\tfrac{y}{x+y}\big)\Lambda_l(x+y+z)\tfrac{\partial^{2i+2k+2l}}{\partial\nu^{2i}\partial y^{2k}\partial z^{2l}}f(0,0,1)\\
&+ \sum_{i=m}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^{i-m}\big(\tfrac{1}{x+y+z}\big)^{2i}(x+y)^{2i-2m}\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_{i-m}\big(\tfrac{y}{x+y}\big)\Lambda_l(1-x-y-z)\tfrac{\partial^{2i+2k+2l}}{\partial\nu^{2i}\partial y^{2k+2l}}f(0,0,0)\\
&+ \sum_{i=m}^{n-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^{i-m}\big(\tfrac{1}{x+y+z}\big)^{2i}(x+y)^{2i-2m}\Lambda_k\big(\tfrac{x+y}{x+y+z}\big)\Lambda_{i-m}\big(\tfrac{y}{x+y}\big)\Lambda_l(x+y+z)\tfrac{\partial^{2i+2k+2l}}{\partial\nu^{2i}\partial y^{2k+2l}}f(0,1,0).
\end{aligned}$$

It is not difficult to see that

$$\frac{\partial^{2p+2m}}{\partial z^{2p}\partial\nu^{2m}}(L_n^T f)(x,y,z)\,\big|_{(0,0,0)} = \frac{\partial^{2p+2m}}{\partial z^{2p}\partial\nu^{2m}}(L_n^T f)(0,y,z)\,\big|_{(0,0,0)}, \quad (33)$$
$$\frac{\partial^{2p+2m}}{\partial z^{2p}\partial\nu^{2m}}(L_n^T f)(x,y,z)\,\big|_{(0,0,1)} = \frac{\partial^{2p+2m}}{\partial z^{2p}\partial\nu^{2m}}(L_n^T f)(0,y,z)\,\big|_{(0,0,1)}, \quad (34)$$

for all m and p chosen such that $m+p$ is not greater than $n-1$. After we make

$i-m\to i$ we obtain

$$\frac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(0,y,z) = \tag{35}$$
$$= \sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\left(\tfrac{1}{y+z}\right)^{2i} y^{2i}\Lambda_k\left(1-\tfrac{y}{y+z}\right)\Lambda_i(0)\Lambda_l(1-y-z)\tfrac{\partial^{2i++2m+2k+2l}}{\partial x^{2k}\partial\nu^{2i+2m}\partial z^{2l}}f(0,0,0)$$
$$+\sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\left(\tfrac{1}{y+z}\right)^{2i} y^{2i}\Lambda_k\left(1-\tfrac{y}{y+z}\right)\Lambda_i(0)\,\Lambda_l(y+z)\tfrac{\partial^{2i++2m+2k+2l}}{\partial x^{2k}\partial\nu^{2i+2m}\partial z^{2l}}f(0,0,1)$$
$$+\sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\left(\tfrac{1}{y+z}\right)^{2i} y^{2i}\Lambda_k\left(\tfrac{y}{y+z}\right)\Lambda_i(0)\,\Lambda_l(1-y-z)f\tfrac{\partial^{2i++2m+2k+2l}}{\partial x^{2l}\partial y^{2k}\partial\nu^{2i+2m}}f(0,0,0)$$
$$+\sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\left(\tfrac{1}{y+z}\right)^{2i} y^{2i}\Lambda_k\left(\tfrac{y}{y+z}\right)\Lambda_i(0)\,\Lambda_l(y+z)\tfrac{\partial^{2i++2m+2k+2l}}{\partial x^{2l}\partial y^{2k}\partial\nu^{2i+2m}}f(1,0,0)$$
$$+\sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\left(\tfrac{1}{y+z}\right)^{2i} y^{2i}\Lambda_k\left(1-\tfrac{y}{y+z}\right)\Lambda_i(1)\,\Lambda_l(1-y-z)\tfrac{\partial^{2i++2m+2k+2l}}{\partial\nu^{2i+2m}\partial y^{2k}\partial z^{2l}}f(0,0,0)$$
$$+\sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\left(\tfrac{1}{y+z}\right)^{2i} y^{2i}\Lambda_k\left(1-\tfrac{y}{y+z}\right)\Lambda_i(1)\,\Lambda_l(y+z)\tfrac{\partial^{2i++2m+2k+2l}}{\partial\nu^{2i+2m}\partial y^{2k}\partial z^{2l}}f(0,0,1)$$
$$+\sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\left(\tfrac{1}{y+z}\right)^{2i} y^{2i}\Lambda_k\left(\tfrac{y}{y+z}\right)\Lambda_i(1)\,\Lambda_l(1-y-z)\tfrac{\partial^{2i++2m+2k+2l}}{\partial\nu^{2i+2m}\partial y^{2k+2l}}f(0,0,0)$$
$$+\sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1} 2^i\left(\tfrac{1}{y+z}\right)^{2i} y^{2i}\Lambda_k\left(\tfrac{y}{y+z}\right)\Lambda_i(1)\,\Lambda_l(y+z)\tfrac{\partial^{2i+2m+2k+2l}}{\partial\nu^{2i+2m}\partial y^{2k+2l}}f(0,1,0).$$

Taking into account the relations (1) we obtain

$$\begin{aligned}\frac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(0,y,z) =& \sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\Lambda_k\left(1-\tfrac{y}{y+z}\right)\Lambda_l(1-y-z)\tfrac{\partial^{2m+2k+2l}}{\partial\nu^{2m}\partial y^{2k}\partial z^{2l}}f(0,0,0)\\ &+\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\Lambda_k\left(1-\tfrac{y}{y+z}\right)\Lambda_l(y+z)\tfrac{\partial^{2m+2k+2l}}{\partial\nu^{2m}\partial y^{2k}\partial z^{2l}}f(0,0,1)\\ &+\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\Lambda_k\left(\tfrac{y}{y+z}\right)\Lambda_l(1-y-z)\tfrac{\partial^{2m+2k+2l}}{\partial\nu^{2m}\partial y^{2k+2l}}f(0,0,0)\\ &+\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\Lambda_k\left(\tfrac{y}{y+z}\right)\Lambda_l(y+z)\tfrac{\partial^{2m+2k+2l}}{\partial\nu^{bv2m}\partial y^{2k+2l}}f(0,1,0).\end{aligned}$$

From [5] we have that

$$\begin{aligned}\tfrac{\partial^{2p}}{\partial z^{2p}}\tfrac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(0,y,z)\big|_{(0,0,0)} &= \tfrac{\partial^{2p}}{\partial z^{2p}}\tfrac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(0,0,z)\big|_{(0,0,0)}\ ,\\ \tfrac{\partial^{2p}}{\partial z^{2p}}\tfrac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(0,y,z)\big|_{(0,0,1)} &= \tfrac{\partial^{2p}}{\partial z^{2p}}\tfrac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(0,0,z)\big|_{(0,0,1)}\ ,\\ \tfrac{\partial^{2p}}{\partial x^{2p}}\tfrac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(x,0,z)\big|_{(1,0,0)} &= \frac{\partial^{2p}}{\partial x^{2p}}\frac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(x,0,0)\big|_{(1,0,0)}\ ,\\ \tfrac{\partial^{2p}}{\partial y^{2p}}\tfrac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(0,y,z)\big|_{(0,1,0)} &= \tfrac{\partial^{2p}}{\partial y^{2p}}\tfrac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(0,y,0)\big|_{(0,1,0)}\ .\end{aligned}\tag{36}$$

We have

$$\begin{aligned}
\frac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(0,0,z) = &\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\Lambda_k(1)\,\Lambda_l(1-z)\frac{\partial^{2m+2k+2l}}{\partial\nu^{2m}\partial y^{2k}\partial z^{2l}}f(0,0,0)\\
&+\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\Lambda_k(1)\,\Lambda_l(z)\frac{\partial^{2m+2k+2l}}{\partial\nu^{2m}\partial y^{2k}\partial z^{2l}}f(0,0,1)\\
&+\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\Lambda_k(0)\,\Lambda_l(1-z)\frac{\partial^{2m+2k+2l}}{\partial\nu^{2m}\partial y^{2k+2l}}f(0,0,0)\\
&+\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\Lambda_k(0)\,\Lambda_l(0)\frac{\partial^{2m+2k+2l}}{\partial\nu^{bv2m}\partial y^{2k+2l}}f(0,1,0).
\end{aligned}$$

Taking into account the relations (1) we obtain

$$\begin{aligned}
\frac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(0,0,z) &= \sum_{l=0}^{n-1}\Lambda_l(1-z)\frac{\partial^{2m+2l}}{\partial\nu^{2m}\partial z^{2l}}f(0,0,0)+\sum_{l=0}^{n-1}\Lambda_l(z)\frac{\partial^{2m+2k+2l}}{\partial\nu^{2m}\partial y^{2k}\partial z^{2l}}f(0,0,1)\\
&= L_n\left(\frac{\partial^{2m}}{\partial\nu^{2m}}f(0,0,\cdot)\right)(z),
\end{aligned}$$

whence, taking into account the Lidstone interpolation conditions (2) and conditions (36), we obtain

$$\begin{aligned}
\frac{\partial^{2p+2m}}{\partial z^{2p}\partial\nu^{2m}}(L_n^T f)(0,0,0) &= \frac{\partial^{2p}}{\partial z^{2p}}L_n\left(\frac{\partial^{2m}}{\partial\nu^{2m}}f(0,0,\cdot)\right)(0) = \frac{\partial^{2p+2m}}{\partial z^{2p}\partial\nu^{2m}}f(0,0,0)\\
\frac{\partial^{2p+2m}}{\partial z^{2p}\partial\nu^{2m}}(L_n^T f)(0,0,1) &= \frac{\partial^{2p}}{\partial z^{2p}}L_n\left(\frac{\partial^{2m}}{\partial\nu^{2m}}f(0,0,\cdot)\right)(1) = \frac{\partial^{2p+2m}}{\partial z^{2p}\partial\nu^{2m}}f(0,0,1).
\end{aligned}$$

We know that

$$\frac{\partial^{2p+2m}}{\partial x^{2p}\partial\nu^{2m}}(L_n^T f)(x,y,z)\Big|_{(1,0,0)} = \frac{\partial^{2p+2m}}{\partial x^{2p}\partial\nu^{2m}}(L_n^T f)(x,0,z)\Big|_{(1,0,0)}.$$

In the same way we calculate $\frac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(x,0,z)$ and making $i-m\to i$, we get

$$\begin{aligned}
&\frac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(x,0,z) =\\
&= \sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1}2^i\big(\tfrac{1}{x+z}\big)^{2i}x^{2i}\Lambda_k\big(1-\tfrac{x}{x+z}\big)\Lambda_i(1)\,\Lambda_l(1-x-z)\frac{\partial^{2i+2m+2k+2l}}{\partial x^{2k}\partial\nu^{2i+2m}\partial z^{2l}}f(0,0,0)\\
&+ \sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1}2^i\big(\tfrac{1}{x+z}\big)^{2i}x^{2i}\Lambda_k\big(1-\tfrac{x}{x+z}\big)\Lambda_i(1)\,\Lambda_l(x+z)\frac{\partial^{2i++2m+2k+2l}}{\partial x^{2k}\partial\nu^{2i+2m}\partial z^{2l}}f(0,0,1)\\
&+ \sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1}2^i\big(\tfrac{1}{x+z}\big)^{2i}x^{2i}\Lambda_k\big(\tfrac{x}{x+z}\big)\Lambda_i(1)\,\Lambda_l(1-x-z)f\frac{\partial^{2i++2m+2k+2l}}{\partial x^{2l}\partial y^{2k}\partial\nu^{2i+2m}}f(0,0,0)\\
&+ \sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1}2^i\big(\tfrac{1}{x+z}\big)^{2i}x^{2i}\Lambda_k\big(\tfrac{x}{x+z}\big)\Lambda_i(1)\,\Lambda_l(x+z)\frac{\partial^{2i++2m+2k+2l}}{\partial x^{2l}\partial y^{2k}\partial\nu^{2i+2m}}f(1,0,0)\\
&+ \sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1}2^i\big(\tfrac{1}{x+z}\big)^{2i}x^{2i}\Lambda_k\big(1-\tfrac{x}{x+z}\big)\Lambda_i(0)\,\Lambda_l(1-x-z)\frac{\partial^{2i+2m+2k+2l}}{\partial\nu^{2i+2m}\partial y^{2k}\partial z^{2l}}f(0,0,0)\\
&+ \sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1}2^i\big(\tfrac{1}{x+z}\big)^{2i}x^{2i}\Lambda_k\big(1-\tfrac{x}{x+z}\big)\Lambda_i(0)\,\Lambda_l(x+z)\frac{\partial^{2i+2m+2k+2l}}{\partial\nu^{2i+2m}\partial y^{2k}\partial z^{2l}}f(0,0,1)\\
&+ \sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1}2^i\big(\tfrac{1}{x+z}\big)^{2i}x^{2i}\Lambda_k\big(\tfrac{x}{x+z}\big)\Lambda_i(0)\,\Lambda_l(1-x-z)\frac{\partial^{2i+2m+2k+2l}}{\partial\nu^{2i+2m}\partial y^{2k+2l}}f(0,0,0)\\
&+ \sum_{i=0}^{n-m-1}\sum_{k=0}^{n-i-1}\sum_{l=0}^{n-i-k-1}2^i\big(\tfrac{1}{x+z}\big)^{2i}x^{2i}\Lambda_k\big(\tfrac{x}{x+z}\big)\Lambda_i(0)\,\Lambda_l(x+z)\frac{\partial^{2i+2m+2k+2l}}{\partial\nu^{2i+2m}\partial y^{2k+2l}}f(0,1,0).
\end{aligned}$$

We obtain

$$\frac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(x,0,0) = \sum_{l=0}^{n-1}\Lambda_l(1-x)\frac{\partial^{2m+2l}}{\partial\nu^{2i+2m}}f(0,0,0) + \sum_{l=0}^{n-1}\Lambda_l(x)\frac{\partial^{2m+2l}}{\partial\nu^{2i+2m}}f(1,0,0)$$
$$= L_n\left(\frac{\partial^{2m}}{\partial\nu^{m}}f(\cdot,0,0)\right)(x),$$

whence, taking into account conditions (2) and (36), we obtain

$$\frac{\partial^{2p+2m}}{\partial x^{2p}\partial\nu^{2m}}(L_n^T f)(1,0,0) = \frac{\partial^{2p}}{\partial x^{2p}}L_n\left(\frac{\partial^{2m}}{\partial\nu^{2m}}f(\cdot,0,0)\right)(1) = \frac{\partial^{2p+2m}}{\partial x^{2p}\partial\nu^{2m}}f(1,0,0). \tag{37}$$

Similarly, we get

$$\frac{\partial^{2p+2m}}{\partial x^{2p}\partial\nu^{2m}}(L_n^T f)(0,0,0) = \frac{\partial^{2p+2m}}{\partial x^{2p}\partial\nu^{2m}}f(0,0,0). \tag{38}$$

We have from (35) the expression of $\frac{\partial^{2m}}{\partial\nu^{2m}}L_n^T(f)(0,y,z)$ and we obtain

$$\frac{\partial^{2m}}{\partial\nu^{2m}}(L_n^T f)(0,y,0) = \sum_{l=0}^{n-1}\Lambda_l(1-y)\frac{\partial^{2m+2l}}{\partial\nu^{2i+2m}}f(0,0,0) + \sum_{l=0}^{n-1}\Lambda_l(y)\frac{\partial^{2m+2l}}{\partial\nu^{2i+2m}}f(0,1,0)$$
$$= L_n\left(\frac{\partial^{2m}}{\partial\nu^{2m}}f(0,\cdot,0)\right)(y).$$

Taking into account that

$$\frac{\partial^{2p+2m}}{\partial y^{2p}\partial\nu^{2m}}(L_n^T f)(x,y,z)\big|_{(0,1,0)} = \frac{\partial^{2p+2m}}{\partial y^{2p}\partial\nu^{2m}}(L_n^T f)(0,y,z)\big|_{(0,1,0)},$$

relations (36) and the Lidstone interpolation conditions (2) we obtain that

$$\frac{\partial^{2p+2m}}{\partial y^{2p}\partial\nu^{2m}}(L_n^T f)(0,1,0) = \frac{\partial^{2p}}{\partial x^{2p}}L_n\big(\frac{\partial^{2m}}{\partial\nu^{2m}}f(0,\cdot,0)\big)(1) = \frac{\partial^{2p+2m}}{\partial y^{2p}\partial\nu^{2m}}f(0,1,0). \tag{39}$$

Analogous, we get

$$\frac{\partial^{2p+2m}}{\partial y^{2p}\partial\nu^{2m}}(L_n^T f)(0,0,0) = \frac{\partial^{2p+2m}}{\partial y^{2p}\partial\nu^{2m}}f(0,0,0). \tag{40}$$

From (33), (34), (37) and (39) the interpolation conditions (14) are satisfied.

Further, following the procedure described in [5], we prove that the interpolation condition (13) also holds. By (28) we have

$$2^j\frac{\partial^{2i}}{\partial x^{2i-2j}\partial\nu^{2j}}f = \sum_{k=0}^{2j}\binom{2j}{k}(-1)^{2j-k}\frac{\partial^{2i}}{\partial x^{2i-k}\partial y^k}f,$$
$$2^j\frac{\partial^{2i}}{\partial y^{2i-2j}\partial\nu^{2j}}f = \sum_{k=0}^{2j}\binom{2j}{k}(-1)^{2j-k}\frac{\partial^{2i}}{\partial x^{k}\partial y^{2i-k}}f.$$

By replacing these results in (38) and (40), we get

$$\begin{cases}\sum\limits_{k=0}^{2j}\binom{2j}{k}(-1)^{2j-k}\frac{\partial^{2i}}{\partial x^{2i-k}\partial y^k}f(0,0,0) = \sum\limits_{k=0}^{2j}\binom{2j}{k}(-1)^{2j-k}\frac{\partial^{2i}}{\partial x^{2i-k}\partial y^k}(L_n^T f)(0,0,0)\\ \sum\limits_{k=0}^{2j}\binom{2j}{k}(-1)^{2j-k}\frac{\partial^{2i}}{\partial x^{k}\partial y^{2i-k}}f(0,0,0) = \sum\limits_{k=0}^{2j}\binom{2j}{k}(-1)^{2j-k}\frac{\partial^{2i}}{\partial x^{k}\partial y^{2i-k}}(L_n^T f)(0,0,0),\end{cases}$$

for $j = 0,...,i$. This is a nonzero $(2i+1)\times(2i+1)$ linear system with the unknowns $\frac{\partial^{2i}}{\partial x^{2i-k}\partial y^k}f(0,0,0)$, $k = 0,...,2i$. The unique solution of this system is

$$\frac{\partial^{2i}}{\partial x^{2i-j}\partial y^j}f(0,0,0) = \frac{\partial^{2i}}{\partial x^{2i-j}\partial y^j}L_n^T(f)(0,0,0),$$

for $j = 0,...,2i$ and for all $i = 0,...,n-1$. So the interpolation condition (13) is proved. ∎

3 Numerical examples

We consider the functions $f : T \to \mathbb{R}$ and $g : T \to \mathbb{R}$ given by

$$f(x, y, z) = x^3 + y^3 + z^3$$

and

$$g(x, y, z) = e^{-(x^2+y^2+z^2)}.$$

In the following tables we present the values of the functions and of the Lidstone approximations at some points of the tetrahedron.

point	f	$L_2^T f$
$(\frac{1}{2}, \frac{1}{2}, 0)$	0.250000	0.250000
$(0, 0, \frac{1}{2})$	0.125000	0.125000
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	0.046875	0.116319
$(\frac{1}{2}, 0, \frac{1}{4})$	0.140625	0.421875
$(\frac{1}{5}, \frac{2}{3}, 0)$	0.304296	0.189425

point	g	$L_2^T g$
$(\frac{1}{2}, \frac{1}{2}, 0)$	0.606531	0.367879
$(0, 0, \frac{1}{2})$	0.778801	0.762955
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	0.829029	0.722597
$(\frac{1}{2}, 0, \frac{1}{4})$	0.731616	0.680667
$(\frac{1}{5}, \frac{2}{3}, 0)$	0.616039	0.516703

Remark 4 *We note that, according to the steps in the proof, in checking the interpolation conditions for example at* $(0, 0, 1)$*, one must take first* $x = 0$ *in the expressions occurring in* L_n^T*, then set* $y = 0$ *and finally* $z = 1$*.*

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Some Initial Value Problems Containing a Large Parameter

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Abstract

We derive the asymptotic behavior of the solution of the problem.

$$w''(t) + \frac{\alpha}{t} w'(t) + B^2 \rho(t)^2 w(t) = 0, \quad t > 0,$$

$$w(0) = 1, \qquad w'(0) = 0,$$

as $B \to \infty$. Here $\alpha > 0$ and $\rho(t) > 0$. We also discuss the asymptotics of the nonlinear Schrödinger-type problem

$$u'' + \frac{\alpha}{t} u' + u^{2p+1} = 0, \qquad t > 0,$$

$$u(0) = \gamma, \qquad u'(0) = 0,$$

as $\gamma \to \infty$.

Key words and phrases. Equations containing a large parameter, WKB, asymptotic matching.

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1 INTRODUCTION

Consider the initial value problem

$$w''(t) + \frac{\alpha}{t} w'(t) + B^2 \rho(t)^2 w(t) = 0, \quad t > 0, \tag{1}$$

$$w(0) = 1, \qquad w'(0) = 0, \tag{2}$$

where $\rho(t)$ is twice continuously differentiable and strictly positive on a given interval $[0, b]$, α is a fixed strictly positive real number, and B is a large parameter.

Such problems arise, e.g., when we are interested in the behavior of the radially symmetric and bounded solution of the multidimensional equation

$$\Delta w + B^2 \rho(r)^2 w = 0$$

(where $r = \sqrt{x_1^2 + \cdots + x_n^2}$), as $B \to \infty$.

In Section 2 of this note we compute the asymptotics of $w(t)$, as $B \to \infty$, using special Liouville-type transformations, asymptotic matching, and the WKB approximation.

In Section 3 we discuss the asymptotics of the solution $u(t)$ of the nonlinear problem

$$u'' + \frac{\alpha}{t} u' + u^{2p+1} = 0, \qquad t > 0,$$

$$u(0) = \gamma, \qquad\qquad u'(0) = 0,$$

as $\gamma \to \infty$. In the case where α is a positive integer, the above equation is equivalent to a multidimensional nonlinear Schrödinger equation with radial symmetry. Using the results of Section 2 we derive heuristically the behavior of the amplitude of the solution, as $\gamma \to \infty$.

The case $\alpha = 2$ which is special and somehow easier to handle has been analyzed in [3].

2 THE LINEAR PROBLEM

Let $b > 0$ be a fixed number and consider the problem (1)–(2), where $\rho(t)$ is a strictly positive function in $C^2[0, b]$. We are interested in the asymptotic behavior of the solution $w(t)$, $t \in [0, b]$, as $B \to \infty$.

2.1 The Case $0 < \alpha < 1$

In this case we introduce the change of variables

$$t = z^\lambda, \qquad \text{where} \quad \lambda = \frac{1}{1-\alpha} > 1, \tag{3}$$

and to make things clear we set

$$v(z) = w(t). \tag{4}$$

In view of the above transformation, a straightforward calculation yields that (1)–(2) is equivalent to

$$v''(z) \; + \; \lambda^2 B^2 z^{2\lambda-2} \rho(z^\lambda)^2 v(z) = 0, \quad z > 0, \tag{5}$$

$$v(0) = 1, \qquad v'(0) = 0. \tag{6}$$

The WKB theory together with asymptotic matching (see, e.g., [1]) implies that in some region I of the form

$$z \gg (1/B)^\sigma, \qquad \sigma > 0 \tag{7}$$

the so-called physical optics approximation to $v(z)$ is

$$v_{\mathrm{I}}(z) \sim [Q(z)]^{-1/4} \; \{C_1 \cos[BS_0(z)] \; + \; C_2 \sin[BS_0(z)]\}, \quad \text{as } B \to \infty, \tag{8}$$

where

$$Q(z) = \lambda^2 z^{2\lambda-2} \rho(z^\lambda)^2, \tag{9}$$

$$S_0(z) = \int_0^z \sqrt{Q(\tau)}\, d\tau = \int_0^z \lambda \tau^{\lambda-1} \rho(\tau^\lambda)\, d\tau = \int_0^{z^\lambda} \rho(\tau)\, d\tau \tag{10}$$

and C_1, C_2 are constants (to be determined). It will be convenient for the sequel to rewrite the approximate formula (8) as follows:

$$\begin{aligned} v_{\mathrm{I}}(z) \quad \sim \quad & \frac{C_1^*}{\sqrt{\lambda z^{\lambda-1} \rho(z^\lambda)}} \cos\left[B \int_0^{z^\lambda} \rho(\tau)\, d\tau + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right] \; + \\ & + \frac{C_2^*}{\sqrt{\lambda z^{\lambda-1} \rho(z^\lambda)}} \sin\left[B \int_0^{z^\lambda} \rho(\tau)\, d\tau + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right], \end{aligned} \tag{11}$$

where z is in region I and C_1^*, C_2^* are constants. The fact that $\rho(t) > 0$, for $t > 0$, guarantees that, for z bounded away from zero, the difference between the exact solution $v(z)$ of (1)–(2) and $v_{\mathrm{I}}(z)$ is of order $1/B$, as $B \to \infty$.

In order to determine region I (i.e. to estimate σ of (7)), we have to check (see [1]) the validity of the following two criteria

$$BS_0 \gg S_1 \gg S_2/B, \quad S_2/B \ll 1, \quad \text{as } \; B \to \infty, \tag{12}$$

where

$$S_1(z) = -\frac{1}{4} \ln[Q(z)], \quad S_2(z) = \pm \int^z \left[\frac{Q''(\tau)}{8(Q(\tau))^{3/2}} - \frac{5(Q'(\tau))^2}{32(Q(\tau))^{5/2}} \right] d\tau. \tag{13}$$

For typographical convenience let us set

$$\rho_0 = \rho(0) > 0.$$

Then, as $z \to 0^+$, i.e. when $z \ll 1$ (see (9), (10), and (13)), we have

$$Q(z) \sim \lambda^2 \rho_0^2 z^{2\lambda-2}, \quad S_0(z) \sim \rho_0 z^\lambda, \quad S_1(z) \sim -\frac{1}{2}\ln(\lambda\rho_0) - \frac{\lambda-1}{2}\ln z$$

and

$$S_2(z) \sim c_1 z^{-\lambda} + c_2,$$

where c_1 and c_2 are constants. Taking into account the above approximations, we infer that the criteria (12) are satisfied for

$$\frac{z}{(\ln z)^{1/\lambda}} \gg (1/B)^{1/\lambda}, \qquad \text{as } B \to \infty.$$

It follows from the above that the region I can be taken as in (7) where σ is any number satisfying

$$\sigma < \frac{1}{\lambda}. \tag{14}$$

Next, we turn to the analysis of the problem (1)–(2) in region II, i.e. for $z > 0$, $z \ll 1$. In this region, the WKB approximation is not valid because Q has a (multiple) zero at 0. Nevertheless, we may solve the approximate problem

$$v''(z) \ + \ B^2\lambda^2\rho_0^2 z^{2\lambda-2} v(z) = 0, \tag{15}$$

$$v(0) = 1, \qquad v'(0) = 0, \tag{16}$$

in terms of Bessel functions. Indeed, the general solution of (15) is (see, e.g., [1])

$$v(z) = \sqrt{z}\left[C J_{1/(2\lambda)}(\rho_0 B z^\lambda) + D J_{-1/(2\lambda)}(\rho_0 B z^\lambda)\right], \qquad z > 0,$$

where C, D are constants.The series expansions of Bessel functions imply that for $z > 0$,

$$J_{1/(2\lambda)}(\rho_0 B z^\lambda) = (\rho_0 B/2)^{1/(2\lambda)} z^{1/2} F(z), \quad J_{-1/(2\lambda)}(\rho_0 B z^\lambda) = (\rho_0 B/2)^{-1/(2\lambda)} z^{-1/2} G(z),$$

where

$$F(z) = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{4}\rho_0^2 B^2 z^{2\lambda}\right)^n}{\Gamma\left(n + \frac{1}{2\lambda} + 1\right)}, \qquad F(0) = \frac{1}{\Gamma\left(1 + \frac{1}{2\lambda}\right)},$$

$\Gamma(\cdot)$ being the Gamma function, and

$$G(z) = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{4}\rho_0^2 B^2 z^{2\lambda}\right)^n}{\Gamma\left(n - \frac{1}{2\lambda} + 1\right)}, \qquad G(0) = \frac{1}{\Gamma\left(1 - \frac{1}{2\lambda}\right)}, \quad G'(0) = 0.$$

Then

$$v(z) = \left(\frac{\rho_0 B}{2}\right)^{1/(2\lambda)} C z F(z) \ + \ (\rho_0 B/2)^{-1/(2\lambda)} D G(z), \qquad z \geq 0,$$

and, hence, by using the initial condition (16) we get

$$C = 0, \qquad D = \left(\frac{\rho_0 B}{2}\right)^{1/(2\lambda)} \Gamma\left(1 - \frac{1}{2\lambda}\right).$$

Consequently,

$$v_{\mathrm{II}}(z) = \left(\frac{\rho_0 B}{2}\right)^{1/(2\lambda)} \Gamma\left(1 - \frac{1}{2\lambda}\right) \sqrt{z} J_{-1/(2\lambda)}(\rho_0 B z^{\lambda}), \qquad z > 0, \quad z \ll 1. \tag{17}$$

It is clear from the above mentioned work that $v_{\mathrm{I}}, v_{\mathrm{II}}$ given by (11), (17) respectively, have a common region of validity (overlap region), namely,

$$(1/B)^{1/\lambda} \ll z \ll 1, \quad B \to \infty.$$

In order to to match these two approximate solutions, we must further approximate them in the overlap region.

First, we consider $v_{\mathrm{I}}(z)$. For $(1/B)^{1/\lambda} \ll z \ll 1$ $(B \to \infty)$, z is "small", so we have

$$\rho(t) \sim \rho_0 t, \qquad \text{as} \quad t \to 0^+,$$

and thus (see (11)),

$$v_{\mathrm{I}}(z) \sim \frac{1}{\sqrt{\rho_0 \lambda z^{\lambda-1}}} \left[C_1^* \cos\left(\rho_0 B z^{\lambda} + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right) + C_2^* \sin\left(\rho_0 B z^{\lambda} + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right)\right] \tag{18}$$

Next, we consider $v_{\mathrm{II}}(z)$. In the overlap region we have $Bz^{\lambda} \to \infty$, as $B \to \infty$, so it is necessary to approximate the Bessel function $J_{-1/(2\lambda)}(\cdot)$ by its leading asymptotic behavior for "large" positive argument. The appropriate formula is (see, e.g., [1])

$$J_{-1/(2\lambda)}(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right), \qquad \text{as} \quad x \to \infty, \tag{19}$$

which implies

$$J_{-1/(2\lambda)}(\rho_0 B z^{\lambda}) \sim \sqrt{\frac{2}{\pi \rho_0 B z^{\lambda}}} \cos\left(\rho_0 B z^{\lambda} + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right), \qquad z \gg (1/B)^{1/\lambda}, \ \ B \to \infty.$$

Now (17) combined with the above asymptotics gives

$$v_{\mathrm{II}}(z) \sim \left(\frac{\rho_0 B}{2}\right)^{1/(2\lambda)} \Gamma\left(1 - \frac{1}{2\lambda}\right) \sqrt{\frac{2}{\pi \rho_0 B z^{\lambda-1}}} \cos\left(\rho_0 B z^{\lambda} + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right), \tag{20}$$

for $1 \gg z \gg (1/B)^{1/\lambda}$.

Requiring that (18), (20) match on the overlap region we obtain

$$C_2^* = 0, \qquad C_1^* = B^{\frac{1}{2\lambda} - \frac{1}{2}} \sqrt{\frac{2\lambda}{\pi} \left(\frac{\rho_0}{2}\right)^{1/\lambda}} \Gamma\left(1 - \frac{1}{2\lambda}\right).$$

In summary, the approximations to $v(z)$ in each of the regions I, II are the following:

$$v_{\mathrm{I}}(z) \sim B^{\frac{1}{2\lambda}-\frac{1}{2}} \sqrt{\frac{2}{\pi}\left(\frac{\rho_0}{2}\right)^{1/\lambda}} \Gamma\left(1-\frac{1}{2\lambda}\right) \frac{1}{\sqrt{z^{\lambda-1}\rho(z^\lambda)}} \cos\left(B\int_0^{z^\lambda} \rho(\tau)\, d\tau + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right),$$

for $z \gg (1/B)^{1/\lambda}$, $B \to \infty$,

and

$$v_{\mathrm{II}}(z) \sim \left(\frac{\rho_0 B}{2}\right)^{1/(2\lambda)} \Gamma\left(1-\frac{1}{2\lambda}\right) \sqrt{z} J_{-1/(2\lambda)}(\rho_0 B z^\lambda), \qquad z > 0, \quad z \ll 1, \quad B \to \infty.$$

Accordingly, in the case $0 < \alpha < 1$ the approximations to the solution $w(t)$ of the original problem (1), (2) are the following:

$$w_{\mathrm{I}}(t) \sim B^{\frac{1}{2\lambda}-\frac{1}{2}} \sqrt{\frac{2}{\pi}\left(\frac{\rho_0}{2}\right)^{1/\lambda}} \Gamma\left(1-\frac{1}{2\lambda}\right) \frac{t^{1/(2\lambda)}}{\sqrt{t\rho(t)}} \cos\left(B\int_0^{t} \rho(\tau)\, d\tau + \frac{\pi}{4\lambda} - \frac{\pi}{4}\right), \tag{21}$$

for $t \gg 1/B$, $B \to \infty$,

and

$$w_{\mathrm{II}}(t) \sim \left(\frac{\rho_0 B}{2}\right)^{1/(2\lambda)} \Gamma\left(1-\frac{1}{2\lambda}\right) t^{1/(2\lambda)} J_{-1/(2\lambda)}(\rho_0 B t), \qquad t > 0, \quad t \ll 1, \quad B \to \infty, \tag{22}$$

where λ is given by (3).

2.2 The Case $\alpha > 1$

If $\alpha > 1$, we use the transformation

$$w(t) = z^{-1} v(z), \qquad t = z^\mu,$$

where

$$\mu = \frac{1}{\alpha - 1} > 1. \tag{23}$$

A straightforward calculation yields that (1)–(2) is equivalent to

$$v''(z) + \mu^2 B^2 z^{2\mu-2} \rho(z^\mu)^2 v(z) = 0, \quad z > 0, \tag{24}$$

$$v(0) = 0, \qquad v'(0) = 1. \tag{25}$$

By using similar arguments as in the case $\alpha < 1$ we get

$$v_{\mathrm{I}}(z) \sim B^{-\frac{1}{2\mu}-\frac{1}{2}} \sqrt{\frac{2}{\pi}\left(\frac{\rho_0}{2}\right)^{-1/\mu}} \Gamma\left(1+\frac{1}{2\mu}\right) \frac{1}{\sqrt{z^{\mu-1}\rho(z^\mu)}} \cos\left(B\int_0^{z^\mu} \rho(\tau)\, d\tau - \frac{\pi}{4\mu} - \frac{\pi}{4}\right),$$

for $z \gg (1/B)^{1/\mu}$, $B \to \infty$,

and

$$v_{\mathrm{II}}(z) \sim \left(\frac{\rho_0 B}{2}\right)^{-1/(2\mu)} \Gamma\left(1 + \frac{1}{2\mu}\right) \sqrt{z} J_{1/(2\mu)}(\rho_0 B z^\mu), \qquad z > 0, \quad z \ll 1, \quad B \to \infty.$$

Thus,

$$w_{\mathrm{I}}(t) \sim B^{-\frac{1}{2\mu}-\frac{1}{2}} \sqrt{\frac{2}{\pi}\left(\frac{\rho_0}{2}\right)^{-1/\mu}} \Gamma\left(1 + \frac{1}{2\mu}\right) \frac{t^{-1/(2\mu)}}{\sqrt{t\rho(t)}} \cos\left(B \int_0^t \rho(\tau)\, d\tau - \frac{\pi}{4\mu} - \frac{\pi}{4}\right),$$

$t > 0$, $t \gg 1/B$, $B \to \infty$,

$$w_{\mathrm{II}}(t) \sim \left(\frac{\rho_0 B}{2}\right)^{-1/(2\mu)} \Gamma\left(1 + \frac{1}{2\mu}\right) t^{-1/(2\mu)} J_{1/(2\mu)}(\rho_0 B t), \qquad t > 0, \quad t \ll 1, \quad B \to \infty,$$

where μ is given by (23).

Since (3) and (23) give

$$-\frac{1}{2\lambda} = \frac{\alpha - 1}{2} \quad (\text{when } 0 < \alpha < 1) \qquad \text{and} \qquad \frac{1}{2\mu} = \frac{\alpha - 1}{2} \quad (\text{when } \alpha > 1),$$

we observe that the last two approximations and (21),(22) have exactly the same form. Consequently, (21), (22) are valid for every $\alpha \neq 1$. We summarize our results in the following theorem.

Theorem. Let $w(t)$ be the solution of the problem (1)–(2), where $t \in [0, b]$ and $\alpha \neq 1$. Then, in region I, i.e. when $t \gg 1/B$, as $B \to \infty$,

$$w_{\mathrm{I}}(t) \sim \frac{1}{B^{\alpha/2}} \sqrt{\frac{2}{\pi}} \left(\frac{\rho_0}{2}\right)^{(1-\alpha)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \frac{1}{t^{\alpha/2}\sqrt{\rho(t)}} \cos\left(B \int_0^t \rho(\tau)\, d\tau - \frac{\pi\alpha}{4}\right), \tag{26}$$

while in region II, i.e. when $t > 0$, $t \ll 1$, as $B \to \infty$,

$$w_{\mathrm{II}}(t) \sim \left(\frac{\rho_0 B}{2}\right)^{(1-\alpha)/2} \Gamma\left(\frac{\alpha+1}{2}\right) t^{(1-\alpha)/2} J_{(\alpha-1)/2}(\rho_0 B t). \tag{27}$$

Remarks. (i) We believe that the above formulas are valid even for $\alpha = 1$.
(ii) Since

$$J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sin x}{\sqrt{x}},$$

if we set $\alpha = 2$ in (26) and (27), the formulas reduce to

$$w(t) \sim \frac{\sin\left[B \int_0^t \rho(\tau) d\tau\right]}{B \rho_0^{1/2} \rho(t)^{1/2} t},$$

valid for all $t \in [0,b]$. This agrees with the formula given in [3].

(iii) It would be nice to have a Langer-type formula for $w(t)$, namely an asymptotic formula which is uniformly valid for all $t \in [0,b]$, as $B \to \infty$. In fact, if

$$\rho'(0) = 0,$$

one can check that the solution $w(t)$ of (1)–(2) satisfies

$$w(t) \sim \left(\frac{\rho_0 B}{2}\right)^{(1-\alpha)/2} \Gamma\left(\frac{\alpha+1}{2}\right) t^{(1-\alpha)/2} \sqrt{\frac{S(t)}{t\rho(t)}} J_{(\alpha-1)/2}\left[BS(t)\right], \qquad \text{for all } t \in [0,b], \tag{28}$$

where

$$S(t) = \int_0^t \rho(\tau)\, d\tau.$$

Unless $\alpha = 2$, one needs the condition $\rho'(0) = 0$ in order for the expression shown in (28) to satisfy the initial condition $w'(0) = 0$.

3 THE NONLINEAR PROBLEM

Let us now consider the nonlinear initial value problem

$$u'' + \frac{\alpha}{t}u' + u^{2p+1} = 0, \qquad t > 0, \tag{29}$$

$$u(0) = \gamma, \qquad\qquad u'(0) = 0, \tag{30}$$

where $p \geq 1$ is a positive integer and $\alpha > 0$ (notice that, if α is an integer, then $u'' + (\alpha/t)u'$ is the $(\alpha+1)$-dimensional radial Laplacian of u, hence (29) is a radially symmetric multidimensional nonlinear Schrödinger equation). Again the boundary conditions must be interpreted in the right way, i.e. as limits when $t \to 0^+$.

Proposition. The problem (29)–(30) has a unique solution for all $t > 0$.

Proof. We first notice that (29)–(30) are equivalent to the integral equation

$$u(t) = \gamma - \int_0^t \frac{t^{1-\alpha}\tau^{\alpha} - \tau}{1-\alpha} u(\tau)^{2p+1} d\tau, \tag{31}$$

where the integrand makes sense even for $\alpha = 1$, since in this case it becomes

$$u(t) = \gamma - \int_0^t \tau\left(\ln t - \ln \tau\right) u(\tau)^{2p+1} d\tau.$$

We must, therefore, look at the map

$$\mathcal{F}[u](t) = \gamma - \int_0^t \frac{t^{1-\alpha}\tau^{\alpha} - \tau}{1-\alpha} u(\tau)^{2p+1} d\tau,$$

mapping $C[0,\varepsilon]$ into itself, for any given $\varepsilon > 0$. It is easy to see that, if ε is chosen sufficiently small, then $\mathcal{F}$ is a contraction, namely

$$\|\mathcal{F}[u] - \mathcal{F}[v]\|_\infty \leq c\,\|u - v\|_\infty\,,$$

where $c < 1$. Hence $\mathcal{F}$ has a unique fixed point u in $C[0,\varepsilon]$ which is the unique solution of (31) in $[0,\varepsilon]$ (and it is automatically smooth). Then the global existence and uniqueness follows by the fact that the energy

$$E(t) = u(t)^{2p+2} + (p+1)u'(t)^2 \tag{32}$$

is decreasing. ■

The solution $u(t)$ of (29)–(30) is highly oscillatory, due to the term u^{2p+1}, but with a decreasing amplitude of oscillation, due to the dissipative term $(\alpha/t)u'$. Using the expression (32) for the energy, we can define the amplitude of oscillation as (the same definition was used in [3])

$$A(t) = E(t)^{1/(2p+2)}. \tag{33}$$

Let $0 = t_0' < t_1' < t_2' < \cdots$ be the (positive) zeros of $u'(t)$. Then

$$A(t_j') = \left|u(t_j')\right| = (-1)^j u(t_j'). \tag{34}$$

In [2], it was shown that for a fixed $j \geq 0$,

$$t_{j+1}' - t_j' = \frac{c_p}{2\gamma^p} + O\left(\gamma^{-2p}\right), \qquad \text{as } \gamma \to \infty, \tag{35}$$

where the constant c_p is given by

$$c_p = 4\sqrt{p+1}\int_0^1 \frac{dx}{\sqrt{1-x^{2p+2}}} = \frac{2\sqrt{\pi}}{\sqrt{(p+1)}}\frac{\Gamma\left(\frac{1}{2p+2}\right)}{\Gamma\left(\frac{p+2}{2p+2}\right)}$$

The problem we want to discuss here is: For a given $b > 0$ determine the (leading) asymptotic behavior of $A(b)$, as $\gamma \to \infty$.

As it was shown in [3], for any $\gamma > 0$ there is an $n = n(\gamma) \geq 0$ such that

$$t_{2n}' \leq b < t_{2n+2}'.$$

We set

$$b^* = t_{2n}' \tag{36}$$

(b^* depends on γ and b; in particular $b^* \leq b$). Thus $u(b^*)$ is a local maximum of $u(t)$ and

$$u(b^*) = A\left(b^*\right). \tag{37}$$

Notice that (35) implies that, as $\gamma \to \infty$,

$$b - b^* = O\left(\gamma^{-p}\right),$$

which, in turn gives (see (33))

$$A(b) - A\left(b^*\right) = O\left(\gamma^{1-p}\right), \tag{38}$$

hence, in order to estimate $A(b)$, it suffices, thanks to (37) and (38), to estimate $u(b^*)$.

By setting

$$u(t) = \gamma u_1(t), \tag{39}$$

(29)–(30) can be written as

$$u_1'' + \frac{\alpha}{t} u_1' + \gamma^{2p} u_1^{2p+1} = 0, \qquad t > 0, \tag{40}$$

$$u_1(0) = 1, \qquad\qquad u'(0) = 0. \tag{41}$$

We propose the following heuristic way to estimate $u_1(b^*)$ as $\gamma \to \infty$. Applying (26) to (40), (41) for $\rho(t) = |u_1(t)|^p$ (hence $\rho_0 = 1$), $B = \gamma^p$, we obtain that for $t > 0$, $t \gg 1/\gamma^p$,

$$u_1(t) \sim \frac{1}{\gamma^{p\alpha/2}} \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right)^{(1-\alpha)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \frac{1}{t^{\alpha/2} |u_1(t)|^{p/2}} \cos\left(\gamma^p \int_0^t |u_1(\tau)|^p \, d\tau - \frac{\pi\alpha}{4}\right),$$

as $\gamma \to \infty$, or, due to (39),

$$u(t) \sim \gamma^{1+[p(1-\alpha)/2]} \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right)^{(1-\alpha)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \frac{1}{t^{\alpha/2} |u(t)|^{p/2}} \cos\left(\int_0^t |u(\tau)|^p \, d\tau - \frac{\pi\alpha}{4}\right). \tag{42}$$

It should be kept in mind that (42) **is valid as long as**

$$A(t) \to \infty, \qquad \text{as } \gamma \to \infty. \tag{43}$$

Formula (42) implies that, under (43),

$$|u(t)|^{(p+2)/2} \sim \gamma^{1+[p(1-\alpha)/2]} \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right)^{(1-\alpha)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \frac{1}{t^{\alpha/2}} \left|\cos\left(\int_0^t |u(\tau)|^p \, d\tau - \frac{\pi\alpha}{4}\right)\right|,$$

or

$$|u(t)| \sim \gamma^{(p+2-p\alpha)/(p+2)} \left(\frac{1}{\pi}\right)^{1/(p+2)} 2^{\alpha/(p+2)} \Gamma\left(\frac{\alpha+1}{2}\right)^{2/(p+2)} \frac{1}{t^{\alpha/(p+2)}} \left|\cos\left(\int_0^t |u(\tau)|^p \, d\tau - \frac{\pi\alpha}{4}\right)\right|^{2/(p+2)},$$

as $\gamma \to \infty$.

Therefore, as $\gamma \to \infty$,

$$A(b) \sim \gamma^{(p+2-p\alpha)/(p+2)} \left(\frac{1}{\pi}\right)^{1/(p+2)} 2^{\alpha/(p+2)} \Gamma\left(\frac{\alpha+1}{2}\right)^{2/(p+2)} \frac{1}{b^{\alpha/(p+2)}},$$

as long as $A(b) \to \infty$, as $\gamma \to \infty$, i.e. when

$$p(\alpha - 1) < 2.$$

If, on the other hand, $p(\alpha - 1) \geq 2$, then

$$A(b) = O(1), \qquad \text{as } \gamma \to \infty.$$

The case $\alpha = 2$ reduces to the statement appeared in [3].

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Monomiality, Lie algebras and Laguerre polynomials

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Abstract: We use the monomiality principle and the Lie algebraic methods to show that the theory of Laguerre polynomials can be framed within a different context, which allows the derivation of previously unknown properties.

MSC: 33C50; 33C10; 33C80

Keywords: Laguerre polynomials; Monomiality principle; Lie algebraic method; Exponential operator.

1. Introduction

According to the principle of monomiality [2], the properties of special polynomials can be deduced from those of ordinary monomials, provided that one can define two operators $\widehat{M}$ and $\widehat{P}$ playing, respectively, the role of multiplicative and derivative operators, for the family of polynomials under study.

In other words if $p_n(x)$ is a set of special polynomials and if

$$\begin{aligned} \widehat{M}p_n(x) &= p_{n+1}(x), \\ \widehat{P}p_n(x) &= np_{n-1}(x), \end{aligned} \tag{1}$$

with

$$p_0(x) = 1, \tag{2}$$

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then we can establish the following correspondence

$$\begin{array}{l} p_n(x) \Leftrightarrow x^n, \\ \widehat{M} \Leftrightarrow x, \\ \widehat{P} \Leftrightarrow \frac{d}{dx}. \end{array} \tag{3}$$

Accordingly, the differential equation satisfied by $p_n(x)$ is given by [2],

$$\widehat{M}\widehat{P}p_n(x) = np_n(x), \tag{4}$$

which can be written in an explicit form once the differential realization of the operators $\widehat{M}$ and $\widehat{P}$ is known.

Let us now consider the case of 2-variable Laguerre polynomials (2VLP), $\mathcal{L}_n(x,y)$ for which the multiplicative and derivative operators are [2],

$$\begin{array}{l} \widehat{M} = y - \widehat{\mathcal{D}}_x^{-1} \\ \widehat{P} = -\frac{\partial}{\partial x}x\frac{\partial}{\partial x}, \end{array} \tag{5}$$

where $\widehat{\mathcal{D}}_x^{-1}$ is the inverse of the derivative operator and is defined in such a way that

$$\widehat{\mathcal{D}}_x^{-n}(1) = \frac{x^n}{n!}. \tag{6}$$

According to the previous prescriptions, we can explicitly derive the polynomials, quasi monomials under the action of the operators (5) as follows

$$\mathcal{L}_n(x,y) = (y - \widehat{\mathcal{D}}_x^{-1})^n = n!\sum_{r=0}^{n} \frac{(-1)^r x^r y^{n-r}}{(r!)^2(n-r)!}. \tag{7}$$

The 2VLP $\mathcal{L}_n(x,y)$ are linked to the ordinary Laguerre polynomials $L_n(x)$ [1], by the relation

$$\begin{array}{l} \mathcal{L}_n(x,y) = y^n L_n(\frac{x}{y}), \\ \mathcal{L}_n(x,1) = L_n(x). \end{array} \tag{8}$$

Also, we observe that

$$\begin{array}{l} \mathcal{L}_n(x,0) = \frac{(-x)^n}{n!}, \\ \mathcal{L}_n(0,y) = y^n. \end{array} \tag{9}$$

In this paper, we combine the Lie algebraic methods and the monomiality principle to study the properties of Laguerre polynomials from a different point of view.

2. Monomiality and Lie algebraic methods

The group $G(0,1)$ [3], generating the ordinary polynomials can be realized by the operators

$$\begin{aligned} &J^{+} = xt, \\ &J^{-} = \tfrac{1}{t}\tfrac{\partial}{\partial x}, \\ &J^{3} = x\tfrac{\partial}{\partial x}, \\ &E = 1, \end{aligned} \tag{10}$$

and the relevant commutation relations are

$$\begin{aligned} &[J^{3}, J^{\pm}] = \pm J^{\pm}, \\ &[J^{+}, J^{-}] = -E. \end{aligned} \tag{11}$$

In view of the correspondence given in Eqs. (3), we can find a different realization of the group $G(0,1)$ in terms of the operators $\widehat{M}$ and $\widehat{P}$, thus for 2VLP $\mathcal{L}_n(x,y)$, we get

$$\begin{aligned} &J^{+} = (y - \widehat{\mathcal{D}}_x^{-1})t, \\ &J^{-} = -\tfrac{1}{t}\tfrac{\partial}{\partial x}x\tfrac{\partial}{\partial x}, \\ &J^{3} = -(y - \widehat{\mathcal{D}}_x^{-1})\tfrac{\partial}{\partial x}x\tfrac{\partial}{\partial x} = (x-y)\tfrac{\partial}{\partial x} - xy\tfrac{\partial^2}{\partial x^2}. \end{aligned} \tag{12}$$

In analogy to the ordinary case we can easily argue that

$$J^{3}\mathcal{L}_n(x,y) = n\mathcal{L}_n(x,y), \tag{13}$$

which, according to Eq. (12), yields the differential equation satisfied by 2VLP $\mathcal{L}_n(x,y)$ in the form

$$\left(xy\frac{\partial^2}{\partial x^2} + (y-x)\frac{\partial}{\partial x} + n\right)\mathcal{L}_n(x,y) = 0. \tag{14}$$

In the realization (12), we have regarded the variable y as a parameter, it is however worth noting that the operator $\frac{\partial}{\partial y}$ plays the same role as the operator $-\frac{\partial}{\partial x}x\frac{\partial}{\partial x}$, therefore the previous results may be complemented with

$$\frac{\partial}{\partial y}\mathcal{L}_n(x,y) = -\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\mathcal{L}_n(x,y). \tag{15}$$

In the following sections, we use the above correspondence to derive old and new properties of Laguerre polynomials.

3. Laguerre polynomials generating functions

Before entering the specific topic of this section let us remind some identities which we will subsequently use. We have introduced the negative derivative operator which can be exploited to define a family of functions known as Tricomi- Bessel(see [2] and the references therein), namely

$$\exp(-\widehat{\mathcal{D}}_x^{-1}) = \sum_{s=0}^{\infty} \frac{(-\widehat{\mathcal{D}}_x^{-1})^s}{s!} = \sum_{s=0}^{\infty} \frac{(-x)^s}{(s!)^2}. \tag{16}$$

The function on the right hand side is the 0^{th}- order Tricomi function defined as follows

$$C_n(x) = \sum_{s=0}^{\infty} \frac{(-x)^s}{s!(n+s)!} = x^{-\frac{n}{2}} J_n(2\sqrt{x}), \tag{17}$$

where $J_n(x)$ are the ordinary cylindrical Bessel functions, and it is also worth noting that

$$\widehat{\mathcal{D}}_x^{-s} C_0(x) = x^s C_s(x). \tag{18}$$

We can get, from the previous discussion and from the quasi monomiality properties of Laguerre polynomials a simple but important result. We note indeed that the generating function of ordinary monomials is defined by

$$\sum_{n=0}^{\infty} \frac{(xt)^n}{n!} = \exp(xt). \tag{19}$$

Replacing x according to the prescription given in Eq. (3), and using Eq. (16), we obtain the following identity [2],

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}_n(x,y) = \exp(yt) C_0(xt), \tag{20}$$

which in view of the link given in Eq. (17), between Tricomi and ordinary Bessel functions is a well documented generating function of Laguerre polynomials [3], derived by means of an operational technique.

Let us now remind that according to the present formalism the associated Laguerre polynomials can be defined as [2],

$$\mathcal{L}_n^{(m)}(x,y) = (1 - y\frac{\partial}{\partial x})^m (y - \widehat{\mathcal{D}}_x^{-1})^n = (1 - y\frac{\partial}{\partial x})^{m+n} \frac{(-x)^n}{n!}, \tag{21}$$

which can be exploited to derive the following relation given in [3],

$$\sum_{l=0}^{\infty} c^{k-l} L_l^{(k-l)}(-bc) x^l = (x+c)^k \exp(bx). \tag{22}$$

Replacing x by $y - \widehat{\mathcal{D}}_x^{-1}$ in Eq. (22), we get

$$\sum_{l=0}^{\infty} c^{k-l} L_l^{(k-l)}(-bc)(y - \widehat{\mathcal{D}}_x^{-1})^l = (y - \widehat{\mathcal{D}}_x^{-1} + c)^k \exp(b(y - \widehat{\mathcal{D}}_x^{-1})), \tag{23}$$

which by means of Eqs. (16,18), yield the following new identity relating the 2VLP $\mathcal{L}_n(x,y)$ and Tricomi functions $C_n(x)$

$$\sum_{l=0}^{\infty} c^{k-l} L_l^{(k-l)}(-bc) \mathcal{L}_l(x,y) = \exp(by) \sum_{s=0}^{k} \binom{k}{s} (-x)^s (y+c)^{k-s} C_s(bx). \tag{24}$$

Again taking $y = 1$ and replacing t by $y - \widehat{\mathcal{D}}_x^{-1}$ and x by z in the generating function (20) and proceeding exactly as above, we obtain

$$\sum_{n=0}^{\infty} L_n(z) \frac{\mathcal{L}_n(x,y)}{n!} = \exp(y) \Pi_0(x,y|z), \tag{25}$$

where

$$\begin{aligned} \Pi_n(x,y|z) &= \sum_{r=0}^{\infty} \tfrac{(-z)^r}{(r!)(n+r)!} \Phi_r(x,y), \\ \Phi_r(x,y) &= \sum_{s=0}^{r} \tbinom{r}{s} y^{r-s} (-x)^s C_s(x). \end{aligned} \tag{26}$$

The function defined by Eq. (26), is similar to the Bessel- Laguerre functions [2]. Eq. (25) can be viewed as a relation between 2VLP $\mathcal{L}_n(x,y)$ and $\Pi_n(x,y|z)$.

Further, we repeat the process with generating relation ([4];p.5 (3.6))

$$\sum_{k=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^{n+r} n!}{k! r! ((n-r)!)^2} (x)^{k+n-r} = \exp(x) L_n(x), \tag{27}$$

to obtain the following relation

$$\sum_{k=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^{n+r} n!}{k! r! ((n-r)!)^2} \mathcal{L}_{n+k-r}(x,y) = \exp(y) L_n(x,y|1), \tag{28}$$

where

$$L_n(x,y|1)=\sum_{s=0}^{n}\frac{(-1)^s n!}{(s!)^2\,(n-s)!}\Phi_s(x,y), \tag{29}$$

is a kind of Laguerre convolution of the Φ_r functions given in Eq. (26).

4. Concluding remarks

The results we have presented in the previous sections yield a clear idea that the combination of Lie algebraic methods and the monomiality principle yields a simple and straightforward approach to get new relations for special polynomials and to define new families of polynomials.

An important point we want to discuss in this section is that whether other formal properties like the orthogonality are preserved by the monomiality correspondence. It is indeed well known that the function [1],

$$\Phi_n(x)=\exp(\frac{-x}{2})L_n(x), \tag{30}$$

provides an orthogonal set, where orthogonal properties can be exploited to derive series expansions of the type

$$\begin{gathered}\exp\left(-ax\right)=\tfrac{1}{(1+a)}\sum_{n=0}^{\infty}\left(\tfrac{a}{1+a}\right)^n L_n\left(x\right),\\ C_0\left(ax\right)=\exp\left(-a\right)\sum_{n=0}^{\infty}\tfrac{a^n}{n!}L_n\left(x\right),\end{gathered} \tag{31}$$

which on the other side, can be viewed as a different restatement of the already quoted generating functions.

It is therefore quite natural to ask whether the function

$$\Psi_n(x,y)=\Phi_n(y-\widehat{\mathcal{D}}_x^{-1})=\exp(\frac{-y}{2})L_n(\frac{-x}{2},y|1), \tag{32}$$

provides an orthogonal set. Even though we can not conclude that $\Psi_n(x,y)$ are orthogonal according to the usual definition, we can conclude that all the expansions of type (32), derived under the assumption of orthogonality, hold under monomiality transform and we find indeed for the first of Eqs. (31).

$$\exp(-a(y-\widehat{\mathcal{D}}_x^{-1}))=\frac{1}{(1+a)}\sum_{n=0}^{\infty}\left(\frac{a}{(1+a)}\right)^n L_n(y-\widehat{\mathcal{D}}_x^{-1}),$$

which yields

$$\exp(-ay)C_0(-ax) = \frac{1}{(1+a)} \sum_{n=0}^{\infty} \left(\frac{a}{(1+a)} \right)^n {}_{\mathcal{L}}\mathcal{L}_n(x,y), \tag{33}$$

where

$${}_{\mathcal{L}}\mathcal{L}_n(x,y) = n! \sum_{r=0}^{n} \frac{(-1)^r \mathcal{L}_r(x,y)}{(r!)^2 (n-r)!}, \tag{34}$$

are the Laguerre-Laguerre polynomials [2]. As to the second of Eqs. (31), we find

$${}_{\mathcal{L}}\mathcal{C}_0(ax, ay) = \exp(-a) \sum_{n=0}^{\infty} \frac{a^n}{n!} \, {}_{\mathcal{L}}\mathcal{L}_n(x,y), \tag{35}$$

where

$${}_{\mathcal{L}}C_n(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r \mathcal{L}_r(x,y)}{r!(n+r)!}, \tag{36}$$

are the Laguerre-Tricomi functions [2].

A further example is provided by the well known expansion

$$x^n = n! \sum_{s=0}^{n} (-1)^s \binom{n}{s} L_s(x), \tag{37}$$

which after monomiality transform yields

$$\mathcal{L}_n(x,y) = n! \sum_{s=0}^{n} (-1)^s \binom{n}{s} {}_{\mathcal{L}}\mathcal{L}_n(x,y). \tag{38}$$

In a forthcoming paper we will discuss how an appropriate modification of the concept of orthogonality can be introduced and how the previous results can be framed within a more formal and rigorous context.

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Generalized System for Relaxed Cocoercive Nonlinear Variational Inequality Problems

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Abstract– Let K_1 and K_2, respectively, be non empty closed convex subsets of real Hilbert spaces H_1 and H_2. The $Approximation-solvability$ of a generalized system of nonlinear relaxed cocoercive variational inequality $(SNVI)$ problems based on the convergence of projection methods is given. The SNVI problem is described as: find an element $(x^*, y^*) \in K_1 \times K_2$ such that

$$< \rho S(x^*, y^*), x - x^* > \geq 0, \ \forall x \in K_1 \text{ and for } \rho > 0,$$

$$< \eta T(x^*, y^*), y - y^* > \geq 0, \ \forall y \in K_2 \text{ and for } \eta > 0,$$

where $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ are nonlinear mappings.

AMS Subject Classification $- 49J40, 65B05, 47H20$
Key Words and Phrases– Relaxed cocoercive mappings, Strongly monotone mappings, Approximation solvability, Projection methods, System of nonlinear variational inequalities.

1. Introduction

Projection/projection - like methods have been applied to the approximation - solvability of problems arising from several fields, including complementarity theory, convex quadratic programming, and variational problems. As a matter of fact, the complementarity problem in mathematical programming is just a special type of variational inequality in finite dimensions. Recently, one of the authors [5] introduced and studied a new system of nonlinear variational inequalities involving strongly monotone mappings in a Hilbert space setting. The study encompasses the classes of cocoercive as well as partially relaxed monotone types of variational inequalities. The notion of the partial relaxed monotonicity is more general than the cocoercivity as well as the strong monotonicity. The obtained results extend/generalize results of Nie et al. [3], Verma [5], and others.For more details on the approximation-solvability of nonlinear varational inequalities, we refer to [1-8].

Let H_1 and H_2 be two real Hilbert spaces with the inner product $< .,. >$ and norm $\|.\|$. Let $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ be any mappings on $K_1 \times K_2$, where K_1 and K_2 are non empty closed convex subsets of H_1 and H_2, respectively. We consider a system of nonlinear variational inequality (abbreviated as SNVI) problems: determine an elements $(x^*, y^*) \in K_1 \times K_2$ such that

$$< \rho S(x^*, y^*), x - x^* > \geq 0 \ \forall x \in K_1 \tag{1}$$

$$< \eta T(x^*, y^*), y - y^* > \geq 0 \ \forall y \in K_2, \tag{2}$$

where $\rho, \eta > 0$.

The SNVI $(1) - (2)$ problem is equivalent to the following projection formulas

$$x^* = P_k[x^* - \rho S(x^*, y^*)] \text{ for } \rho > 0$$

$$y^* = Q_k[y^* - \eta T(x^*, y^*)] \text{ for } \eta > 0,$$

where P_k is the projection of H_1 onto K_1 and Q_K is the projection of H_2 onto K_2.

We note that the SNVI $(1) - (2)$ problem extends the NVI problem: determine an element $x^* \in K_1$ such that

$$< S(x^*), x - x^* > \geq 0, \ \forall x \in K_1. \tag{3}$$

Also, we note that the SNVI $(1) - (2)$ problem is equivalent to a system of nonlinear complementarities (abbreviated as SNC): find an element $(x^*, y^*) \in K_1 \times K_2$ such that $S(x^*, y^*) \in K_1^*$, $T(x^*, y^*) \in K_2^*$ and

$$< \rho S(x^*, y^*), x^* >= 0 \text{ for } \rho > 0, \tag{4}$$

$$< \eta T(x^*, y^*), y^* >= 0 \text{ for } \eta > 0, \tag{5}$$

where K_1^* and K_2^*, respectively, are polar cones to K_1 and K_2 defined by

$$K_1^* = \{f \in H_1 :< f, x > \geq 0, \ \forall x \in K_1\}.$$
$$K_2^* = \{g \in H_2 :< g, y > \geq 0, \ \forall y \in K_2\}.$$

Now, we recall some auxiliary results and notions crucial to the problem on hand.

Lemma 1.1[3]. For an element $z \in H$, we have

$$x \in K \text{ and } < x - z, y - x > \geq 0, \ \forall y \in K \text{ if and only if } x = P_k(z).$$

A mapping $T : H \to H$ from a Hilbert space H into H is called monotone if $< T(x) - T(y), x - y > \geq 0 \forall x, y \in H$. The mapping T is $(r) - strongly$ monotone if for each $x, y \in H$, we have

$$< T(x) - T(y), x - y > \geq r||x - y||^2 \text{ for a constant } r > 0.$$

This implies that $\|T(x) - T(y)\| \geq r\|x - y\|$, that is, T is (r)-expansive, and when $r = 1$, it is expansive. The mapping T is called (s)-Lipschitz continuous (or Lipschitzian) if there exists a constant $s \geq 0$ such that $\|T(x) - T(y)\| \leq s\|x - y\|, \ \forall x, y \in H$. T is called (μ)-cocoercive if for each $x, y \in H$, we have

$$< T(x) - T(y), x - y > \geq \mu||T(x) - T(y)||^2 \text{ for a constant } \mu > 0.$$

Clearly, every (μ)-cocoercive mapping T is $(\frac{1}{\mu})$-Lipschitz continuous. We can easily see that the following implications on monotonicity, strong monotonicity and expansiveness hold:

$$\begin{array}{ccc} \text{strong monotonicity} & \Rightarrow & \text{expansiveness} \\ \Downarrow & & \\ \text{monotonicity} & & \end{array}$$

T is called relaxed (γ)-cocoercive if there exists a constant $\gamma > 0$ such that

$$< T(x) - T(y), x - y > \geq (-\gamma) \|T(x) - T(y)\|^2, \ \forall x, y \in H.$$

T is said to be (r)-strongly pseudomonotone if there exists a positive constant r such that

$$< T(y), x - y > \geq 0 \Rightarrow < T(x), x - y > \geq r \|x - y\|^2, \ \forall x, y \in H.$$

T is said to be relaxed (γ, r)-cocoercive if there exist constants γ,$r > 0$ such that

$$< T(x) - T(y), x - y > \geq (-\gamma) \|T(x) - T(y)\|^2 + r \|x - y\|^2.$$

Clearly, it implies that

$$< T(x) - T(y), x - y > \geq (-\gamma) \|T(x) - T(y)\|^2,$$

that is T is relaxed (γ)-cocoercive.
T is said to be relaxed (γ, r)-pseudococoercive if there exist positive constants γ and r such that

$$< T(y), x - y > \geq 0 \Rightarrow < T(x), x - y > \geq (-\gamma) \|T(x) - T(y)\|^2 + r \|x - y\|^2, \forall x, y \in H.$$

Thus, we have following implications:

$$\begin{array}{ccc} (r)\text{-strong monotonicity} & \Rightarrow & \text{strong } (r)\text{-pseudomonotonicity} \\ \Downarrow & & \\ \text{relaxed } (\gamma, r)\text{-cocoercivity} & & \\ \Downarrow & & \\ \text{relaxed } (\gamma, r)\text{-pseudococoercivity} & & \end{array}$$

2. Projection Methods

This section deals with the convergence of projection methods in the context of the approximation- solvability of the SNVI $(1) - (2)$ problem.

Algorithm 2.1. For an arbitrarily chosen initial point $(x^0, y^0) \in K_1 \times K_2$, compute the sequences $\{x^k\}$ and $\{y^k\}$ such that

$$x^{k+1} = (1 - a^k)x^k + a^k P_k[x^k - \rho S(x^k, y^k)]$$
$$y^{k+1} = (1 - a^k)y^k + a^k Q_K[y^k - \eta T(x^k, y^k)],$$

where P_K is the projection of H_1 onto K_1, Q_K is the projection of H_2 onto K_2, ρ, $\eta > 0$ are constants and, $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ are any two mappings
We now present, based on Algorithm 2.1, the approximation solvability of the SNVI $(1) - (2)$ problem involving relaxed cocoercive and Lipschitz continuous mappings in Hilbert space settings.

Theorem 2.1. Let H_1 and H_2 be two real Hilbert spaces and, K_1 and K_2, respectively, be nonempty closed convex subsets of H_1 and H_2. Let $S : K_1 \times K_2 \to H_1$ be relaxed $(\gamma, r)-cocoercive$ and $(\mu)-Lipschitz$ continuous in the first variable and let S be $(\nu)-Lipschitz$ continuous in the second variable. Let $T : K_1 \times K_2 \to H_2$ be relaxed $(\lambda, s) - cocoercive$ and $(\beta) - Lipschitz$ continuous in the second variable and let T be $(\tau)-Lipschitz$ continuous in the first variable. Let $\|.\|^*$ denote the norm on $H_1 \times H_2$ defined by

$$\|(x, y)\|^* = (\|x\| + \|y\|)\, \forall (x, y) \in H_1 \times H_2.$$

In addition, let $\delta = max\{\theta + \eta\tau, \sigma + \rho\nu\}$, where

$$\theta + \eta\tau = \sqrt{1 - 2\rho r + 2\rho\gamma\mu^2 + \rho^2\mu^2} + \eta\tau < 1$$
$$\sigma + \rho\nu = \sqrt{1 - 2\eta r + 2\eta\lambda\beta^2 + \eta^2\beta^2} + \rho\nu < 1.$$

Suppose that $(x^*, y^*) \in K_1 \times K_2$ forms a solution to the SNVI $(1) - (2)$ problem, the sequences $\{x^k\}$ and $\{y^k\}$ are generated by Algorithm 2.1, and $0 \leq a^k \leq 1$ and $\sum_{k=0}^{\infty} a^k = \infty$.
Then the sequence $\{(x^k, y^k)\}$ converges to (x^*, y^*).

Proof. Since $(x^*, y^*) \in K_1 \times K_2$ forms a solution to the SNVI $(1) - (2)$ problem, it follows that

$$x^* = P_K[x^* - \rho S(x^*, y^*)] \text{ and } y^* = Q_K[x^* - \eta T(x^*, y^*)].$$

Applying Algorithm 2.1, we have

$$\begin{aligned}
&\left\|x^{k+1} - x^*\right\| \\
&\quad = \|(1 - a^k)x^k + a^k P_K[x^k - \rho S(x^k, y^k)] \\
&\quad - (1 - a^k)x^* - a^k P_K[x^* - \rho S(x^*, y^*)]\| \\
&\quad \leq (1 - a^k)\left\|x^k - x^*\right\| \\
&\quad + a^k \left\|P_K[x^k - \rho S(x^k, y^k)] - P_K[x^* - \rho S(x^*, y^*)]\right\| \\
&\quad \leq (1 - a^k)\|x^k - x^*\| + a^k\|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k) \\
&\quad + S(x^*, y^k) - S(x^*, y^*)]\| \\
&\quad \leq (1 - a^k)\|x^k - x^*\| + a^k\|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k)]\| \\
&\quad + \rho\|[S(x^*, y^k) - S(x^*, y^*)]\|. \qquad (6)
\end{aligned}$$

Since S is relaxed $(\gamma, r) - cocoercive$ and $(\mu) - Lipschitz$ continuous in the first variable, and S is $(\nu) - Lipschitz$ continuous in the second variable, we have

$$\begin{aligned}
&\|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k)]\|^2 \\
&\qquad = \|x - x^*\|^2 - 2\rho\langle S(x^k, y^k) - S(x^*, y^k), x^k - x^*\rangle \\
&\qquad + \rho^2\|S(x^k, y^k) - S(x^*, y^k)\|^2 \\
&\qquad \leq \|x^k - x^*\|^2 - 2\rho r\|x^k - x^*\|^2 \\
&\qquad + 2\rho\gamma\|S(x^k, y^k) - S(x^*, y^k)\|^2 \\
&\qquad + \rho^2\mu^2\|x^k - x^*\|^2 \\
&\qquad \leq \|x^k - x^*\|^2 - 2\rho r\|x^k - x^*\|^2 \\
&\qquad + 2\rho\gamma\mu^2\|x^k - x^*\|^2 \\
&\qquad + \rho^2\mu^2\|x^k - x^*\|^2 \\
&\qquad = [1 - 2\rho r + 2\rho\gamma\mu^2 + \rho^2\mu^2]\|x^k - x^*\|^2.
\end{aligned}$$

As a result, we have

$$\begin{aligned}
\left\|x^{k+1} - x^*\right\| \leq (1 - a^k)\|x^k - x^*\| + a^k\theta\|x^k - x^*\| \\
+\rho\nu\|y^k - y^*\|,
\end{aligned} \tag{7}$$

where $\theta = \sqrt{1 - 2\rho r + 2\rho\gamma\mu^2 + \rho^2\mu^2}$.
Similarly, we have

$$\begin{aligned}
\left\|y^{k+1} - y^*\right\| \leq (1 - a^k)\|y^k - y^*\| + a^k\sigma\|y^k - y^*\| \\
+\eta\tau\|x^k - x^*\|,
\end{aligned} \tag{8}$$

where $\sigma = \sqrt{1 - 2\eta r + 2\eta\lambda\beta^2 + \eta^2\beta^2}$.
It follows from (7) and (8) that

$$\begin{aligned}
\|x^{k+1} - x^*\| + \|y^{k+1} - y^*\| &\leq (1 - a^k)\|x^k - x^*\| + a^k\theta\|x^k - x^*\| \\
+\eta\tau\|x^k - x^*\| &+ (1 - a^k)\|y^k - y^*\| \\
&+ a^k\sigma\|y^k - y^*\| + \rho\nu\|y^k - y^*\| \\
&= [1 - (1 - \delta)a^k](\|x^k - x^*\| + \|y^k - y^*\|) \\
&\leq \prod_{j=0}^{k}[1 - (1 - \delta)a^j](\|x^0 - x^*\| + \|y^0 - y^*\|),
\end{aligned}$$

where $\delta = max\{\theta + \eta\tau, \sigma + \rho\nu\}$ and $H_1 \times H_2$ is a Banach space under the norm $\|.\|^*$.

Since $0 < \delta < 1$ and $\sum_{k=0}^{\infty} a^k$ is divergent, it implies in light of [6] that

$$lim_{k \to \infty} \prod_{j=0}^{k} [1 - (1 - \delta) a^j] = 0.$$

Note that with the given conditions, it is clear that the product in limit about is positive for all k. On the other hand, observing that $f(t) = -t + ln(1-t)$ satisfies $f(0) = 0$ and $f' > 0$ for $0 < t < 1$, yields

$$ln[1 - (1 - \delta) a^k] < -(1 - \delta) a^k.$$

Hence,

$$\prod_{j=0}^{k} [1 - (1 - \delta) a^j] = exp \sum_{j=0}^{k} [1 - (1 - \delta) a^j] < exp(-(1 - \delta) \sum_{j=0}^{k} a^j) \to 0,$$

by virtue of the divergence of $\sum_{k=0}^{\infty} a^k$.
Therefore,

$$\|x^{k+1} - x^*\| + \|y^{k+1} - y^*\| \to 0.$$

Consequently, the sequence $\{(x^k, y^k)\}$ converges strongly to (x^*, y^*), a solution to the $SNVI\,(1) - (2)$ problem. This completes the proof.

Corollary 2.1. Let H_1 and H_2 be two real Hilbert spaces and, K_1 and K_2, respectively, be nonempty closed convex subsets of H_1 and H_2. Let $S : K_1 \times K_2 \to H_1$ be $(r) - strongly$ monotone and $(\mu) - Lipschitz$ continuous in the first variable and let S be $(\nu) - Lipschitz$ continuous in the second variable. Let $T : K_1 \times K_2 \to H_2$ be $(s) - strongly$ monotone and $(\beta) - Lipschitz$ continuous in the second variable and let T be $(\tau) - Lipschitz$ continuous in the first variable. Let $\|.\|^*$ denote the norm on $H_1 \times H_2$ defined by

$$\|(x, y)\|^* = (\|x\| + \|y\|)\, \forall (x, y) \in H_1 \times H_2.$$

In addition, let $\delta = max\{\theta + \eta\tau, \sigma + \rho\nu\}$, where

$$\theta + \eta\tau = \sqrt{1 - 2\rho r + \rho^2 \mu^2} + \eta\tau < 1$$

$$\sigma + \rho\nu = \sqrt{1 - 2\eta s + \eta^2 \beta^2} + \rho\nu < 1.$$

Suppose that $(x^*, y^*) \in K_1 \times K_2$ forms a solution to the SNVI $(1) - (2)$ problem, the sequences $\{x^k\}$ and $\{y^k\}$ are generated by Algorithm 2.1, and $0 \leq a^k \leq 1$

and $\sum_{k=0}^{\infty} a^k = \infty$.

Then the sequence $\{(x^k, y^k)\}$ converges to (x^*, y^*).

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ON THE SECOND ORDER DISCONTINUOUS DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, an existence theorem for the second order ordinary differential inclusions is proved without the continuity of multi-functions.

1. Introduction

Let $\mathbb{R}$ be the real line and let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$. Consider the second order differential inclusion (in short, DI):

$$\left.\begin{aligned} &x''(t) \in F(t, x(t), x'(t)), \text{ a.e. } t \in J, \\ &x^{(i)}(0) = x_i \in \mathbb{R}, \ i = 0, 1, \end{aligned}\right\} \tag{1.1}$$

where $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ and $\mathcal{P}_p(\mathbb{R})$ is the class of all non-empty subsets of $\mathbb{R}$ with property p.

By a solution of the DI (1.1) we mean a function $x \in AC^1(J, \mathbb{R})$ that satisfies $x''(t) = v(t)$ for some $v \in L^1(J, \mathbb{R})$ satisfying $v(t) \in F(t, x(t), x'(t))$ for a.e. $t \in J$ and $x^{(i)}(0) = x_i \in \mathbb{R}$ for $i = 0, 1$, where $AC^1(J, \mathbb{R})$ is the space of continuous real-valued functions whose first derivative exists and is absolutely continuous on J.

The DI (1.1) has already been studied in the literature for the existence results under different continuity conditions of F. The existence theorem for DI (1.1) for upper semi-continuous multi-valued function F is proved in Benchohra [3]. When F has closed convex values and is lower semi-continuous, the existence results of DI (1.1) reduce to existence results of the ordinary second order differential equations

$$\left.\begin{aligned} &x''(t) = f(t, x(t), x'(t)), \text{ a.e. } t \in J, \\ &x^{(i)}(0) = x_i \in \mathbb{R}, \ i = 0, 1, \end{aligned}\right\} \tag{1.2}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$f(t, x(t), x'(t)) \in F(t, x(t), x'(t)), \quad \text{a.e.} \quad t \in J.$$

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The case of discontinuous multi-valued function F has been treated in Dhage et al. [7] under monotonic conditions of F and proved the existence of extremal solutions using a lattice fixed point theorem of Dhage and O'Regan [8] in complete lattices. Note that the monotonic condition used in the above paper is very strong and not every Banach space is a complete lattice. These facts motivated us to pursue the study of the present paper.

In this paper, we prove the existence results for the DI (1.1) under a monotonic condition different from that presented in Dhage et al. [7].

2. Auxiliary results

We equip the space $X = AC^1(J, \mathbb{R})$ with the norm $\|\cdot\|$ and the order relation $\leq$ defined by

$$\|x\| = \max\left\{\sup_{t\in J}|x(t)|, \sup_{t\in J}|x'(t)|\right\}$$

and

$$x \leq y \iff x(t) \leq y(t) \wedge x'(t) \leq y'(t), \quad t \in J,$$

respectively.

Let $A, B \in \mathcal{P}_p(X)$. Then by $A \overset{i}{\leq} B$ we mean that, for every $a \in A$, there exists a $b \in B$ such that $a \leq b$. Again, $a \overset{d}{\leq} B$ means that, for each $b \in B$, there exists a $a \in A$ such that $a \leq b$. Further, we have $A \overset{id}{\leq} B \iff A \overset{i}{\leq} B$ and $A \overset{d}{\leq} B$. Finally, $A \leq B$ implies that $a \leq b$ for all $a \in A$ and $b \in B$ (see Dhage [6] and references therein for more details).

Definition 2.1. A mapping $Q : X \to \mathcal{P}_p(X)$ is called right monotone increasing (resp., left monotone increasing) if $Qx \overset{i}{\leq} Qy$ (resp., $Qx \overset{d}{\leq} Qy$) for all $x, y \in X$ with $x \leq y$. Similarly, Q is called monotone increasing if it is left as well as right monotone increasing on X.

We need the following fixed point theorem in the sequel.

Theorem 2.1. ([5]) *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$ be a right monotone increasing multi-valued mapping. If every sequence $\{y_n\} \subset \bigcup Q([a, b])$ defined by $y_n \in Qx_n$ for $n \in \mathbb{N}$ has a cluster point whenever $\{x_n\}$ is a monotone increasing sequence in $[a, b]$, then Q has a fixed point.*

3. The Existence Results

We need the following definitions in the sequel.

Definition 3.1. A multi-valued function $\beta : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ is called L^1-Chandrabhan if

(i) $t \mapsto \beta(t, x, y)$ is measurable for each $x.y \in \mathbb{R}$,

(ii) $\beta(t,x,y)$ is right monotone increasing in x and y almost everywhere for $t \in J$,

(iii) for each real number $r > 0$, there exists a function $h_r \in L^1(J, \mathbb{R})$ such that

$$\|\beta(t,x,y)\|_{\mathcal{P}} = \sup\{|u| : u \in \beta(t,x,y)\} \leq h_r(t), \text{ a.e. } t \in J,$$

for all $x, y \in \mathbb{R}$ with $|x| \leq r$, $|y| \leq r$.

Denote

$$S_F^1(x) = \{u \in L^1(J, \mathbb{R}) : u(t) \in F(t, x(t), x'(t)), \text{ a.e. } t \in J\}$$

for some $x \in AC^1(J, \mathbb{R})$. The integral of the multi-valued function F is defined as

$$\int_0^t F(s, x(s), x'(t))\, ds = \left\{ \int_0^t v(s)\, ds : v \in S_F^1(x) \right\}.$$

Definition 3.2. A function $a \in AC^1(J, \mathbb{R})$ is called a lower solution of the DI (1.1) if, for all $v \in S_F^1(a)$,

$$a''(t) \leq v(t), \text{ a.e. } t \in I,$$
$$a(0) \leq x_0, \ a'(0) \leq x_1.$$

Similarly, an upper solution b to DI (1.1) is defined.

We consider the following hypotheses in the sequel.

(H_1) $F(t,x,y)$ is closed and bounded for each $t \in J$ and $x, y \in \mathbb{R}$.
(H_2) $S_F^1(x) \neq \emptyset$ for all $x \in AC^1(J, \mathbb{R})$.
(H_3) F is L^1-Chandrabhan.
(H_4) DI (1.1) has a lower solution a and an upper solution b with $a \leq b$.

Hypotheses (H_1)-(H_2) are common in the literature. Some nice sufficient conditions for guarantying (H_2) appear in Deimling [4], Lasota and Opial [10]. A mild hypothesis (H_4) has been used in Halidias and Papageorgiou [9]. Hypothesis (H_3) is relatively new to the literature, but the special forms have been appeared in the works of several authors (see Dhage [5, 6] and references therein).

Theorem 3.1. *Assume that* (H_1)-(H_4) *hold. Then the DI (1.1) has a solution in* $[a,b]$.

Proof. Let $X = AC^1(J, \mathbb{R})$ and define an order interval $[a,b]$ in X which is well defined in view of hypothesis (H_4). Now, the DI (1.1) is equivalent to the integral inclusion

$$x(t) \in x_0 + x_1 t + \int_0^t (t-s) F(s, x(s), x'(s))\, ds, \quad t \in J. \tag{3.1}$$

Define a multi-valued operator $Q : [a, b] \to \mathcal{P}_p([a, b])$ by

$$\begin{aligned} Qx &= \left\{ u \in X : u(t) = x_0 + x_1 t + \int_0^t (t-s)v(s)\,ds,\ v \in S_F^1(x) \right\} \\ &= (\mathcal{L} \circ S_F^1)(x), \end{aligned} \tag{3.2}$$

where $\mathcal{L} : L^1 J, \mathrm{I\!R}) \to AC^1(J, \mathrm{I\!R})$ is a continuous operator defined by

$$\mathcal{L}x(t) = x_0 + x_1 t + \int_0^t (t-s)x(s)\,ds.$$

Clearly the operator Q is well defined in view of hypothesis (H_2). We shall show that Q satisfies all the conditions of Theorem 2.1.

Step I: First, we show that Q has compact values on $[a, b]$. It is enough to prove that the Nemetskii operator S_F^1 has compact values on $[a, b]$. This is achieved by showing that every sequence in $S_F^1(x)$ has a convergent subsequence for each $x \in [a, b]$. Let $\{v_n\}$ be a sequence in $S_F^1(x)$. Then $\{v_n(t)\} \in F(t, x(t), x'(t))$ for a.e. $t \in J$. Since $F(t, x(t), x'(t))$ is closed and bounded, it is compact. Hence $\{v_n(t)\}$ has a convergent subsequence converging to a point $v(t) \in F(t, x(t), x'(t))$ for a.e. $t \in J$. Thus there is a subsequence S of $\mathrm{I\!N}$ (: the set of natural numbers) such that $v_n(t) \to v(t)$ in measure as $n \to \infty$ through S. From the continuity of $\mathcal{L}$, it follows that $\mathcal{L}v_n(t) \to \mathcal{L}v(t)$ pointwise on J as $n \to \infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{L}v_n\}$ is an equi-continuous sequence. In fact, let $t,\, \tau \in J$. Then we have

$$\begin{aligned} &|\mathcal{L}v_n(t) - \mathcal{L}v_n(\tau)| \\ &\le |x_1|\,|t-\tau| + \left| \int_0^t (t-s)v_n(s)\,ds - \int_0^\tau (\tau-s)v_n(s)\,ds \right| \\ &\le |x_1||t-\tau| + \left| \int_0^t (t-s)v_n(s)\,ds - \int_0^t (\tau-s)v_n(s)\,ds \right| \\ &\quad + \left| \int_0^t (\tau-s)v_n(s)\,ds - \int_0^\tau (\tau-s)v_n(s)\,ds \right| \\ &\le |x_1||t-\tau| + \left| \int_0^t (t-\tau)v_n(s)\,ds \right| + \left| \int_\tau^t (\tau-s)v_n(s)\,ds \right| \\ &\le |x_1|\,|t-\tau| + \left| \int_0^T |t-\tau||v_n(s)|\,ds \right| + T\left| \int_\tau^t |v_n(s)|\,ds \right|. \end{aligned} \tag{3.3}$$

Again, it follows that

$$\begin{aligned} &|(\mathcal{L}v_n)'(t) - (\mathcal{L}v_n)'(\tau)| \\ &\le \left| \int_0^t v_n(s)\,ds - \int_0^\tau v_n(s)\,ds \right| \le \left| \int_\tau^t |v_n(s)|\,ds \right|. \end{aligned} \tag{3.4}$$

Since $v_n \in L^1(J, \mathbb{R})$, the right hand sides of (3.3) and (3.4) tends to 0 as $t \to \tau$. Hence $\{\mathcal{L}v_n\}$ is equi-continuous and so an application of the Arzelá-Ascoli theorem implies that there is a uniformly convergent subsequence. We then have $\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L} \circ S_F^1)(x)$ as $j \to \infty$ and so $(\mathcal{L} \circ S_F^1)(x)$ is compact. Therefore, Q is a compact-valued multi-valued operator on $[a, b]$.

Step II: Secondly, we show that Q is right monotone increasing and maps $[a, b]$ into itself. Let $x, y \in [a, b]$ be such that $x \leq y$. Since $F(t, x, y)$ is right monotone increasing in x and y, one has $F(t, x, x') \overset{i}{\leq} F(t, y, y')$ for all $t \in J$ and $x \in AC^1(J, \mathbb{R})$. As a result, we have that $S_F^1(x) \overset{i}{\leq} S_F^1(y)$. Hence $Q(x) \overset{i}{\leq} Q(y)$. From (H_3) it follows that $a \leq Qa$ and $Qb \leq b$. Now, Q is right monotone increasing and so we have

$$a \leq Qa \overset{i}{\leq} Qx \overset{i}{\leq} Qb \leq b$$

for all $x \in [a, b]$. Hence Q defines a multi-valued operator $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$.

Step III: Finally, let $\{x_n\}$ be a monotone increasing sequence in $[a, b]$ and let $\{y_n\}$ be a sequence in $Q([a, b])$ defined by $y_n \in Qx_n$ for $n \in \mathbb{N}$.

We shall show that $\{y_n\}$ has a cluster point. This is achieved by showing that $\{x_n\}$ is uniformly bounded and equi-continuous sequence.

First, we show that $\{y_n\}$ is uniformly bounded sequence. By definition of $\{y_n\}$, there is a $v_n \in S_F^1(x_n)$ such that

$$y_n(t) = x_0 + x_1 t + \int_0^t (t - s) v_n(s)\, ds, \ t \in J.$$

Therefore, we have

$$\begin{aligned} |y_n(t)| &\leq |x_0| + |x_1 t| + \int_0^t |t - s||v_n(s)|\, ds \\ &\leq |x_0| + |x_1|T + T \int_0^t \|F(s, x_n(s), x_n'(s))\|\, ds \\ &\leq |x_0| + |x_1|T + T \int_0^T h_r(s)\, ds \\ &\leq |x_0| + \big(|x_1| + \|h_r(s)\|_{L^1}\big)T \end{aligned}$$

and

$$\begin{aligned} |y_n'(t)| &\leq |x_1| + \int_0^t |v_n(s)|\, ds \\ &\leq |x_1| + \int_0^t \|F(s, x_n(s), x_n'(s))\|\, ds \\ &\leq |x_1| + \int_0^T h_r(s)\, ds \end{aligned}$$

$$\leq |x_1| + \|h_r(s)\|_{L^1}$$

for all $t \in J$, where $r = \|a\| + \|b\|$. Therefor, it follows that

$$\begin{aligned}
\|y_n\| &= \max\{\sup_{t\in J} |y_n(t)|\,,\, \sup_{t\in J} |y_n'(t)|\} \\
&\leq \max\{|x_0| + \big(|x_1| + \|h_r(s)\|_{L^1}\big)T\,,\, |x_1| + \|h_r(s)\|_{L^1}\} \\
&= M,
\end{aligned}$$

where $M = |x_0| + (1+T)\big(|x_1| + \|h_r(s)\|_{L^1}\big)$. This shows that $\{y_n\}$ is a uniformly bounded sequence in $Q([a,b])$.

Next, we show that $\{y_n\}$ is an equi-continuous sequence in $Q([a,b])$. Let $t, \tau \in J$. Then we have

$$\begin{aligned}
&|y_n(t) - y_n(\tau)| \\
&\leq |tx_1 - \tau x_1| + \Big|\int_0^t (t-s)v_n(s)\,ds - \int_0^\tau (\tau - s)v_n(s)\,ds\Big| \\
&\leq |x_1||t-\tau| + \Big|\int_0^t (t-s)v_n(s)\,ds - \int_0^t (\tau-s)v_n(s)\,ds \\
&\quad + \int_0^t (\tau-s)v_n(s)\,ds - \int_0^\tau (\tau-s)v_n(s)\,ds\Big| \\
&\leq |x_1||t-\tau| + \Big|\int_0^t (t-\tau)v_n(s)\,ds\Big| + \Big|\int_\tau^t (\tau-s)v_n(s)\,ds\Big| \\
&\leq |x_1|\,|t-\tau| + \Big|\int_0^T |t-\tau||v_n(s)|\,ds\Big| + \Big|\int_\tau^t |\tau-s||v_n(s)|\,ds\Big| \\
&\leq |x_1|\,|t-\tau| + \int_0^T |t-\tau|h_r(s)\,ds + \Big|\int_\tau^t |\tau-s|h_r(s)\,ds\Big| \\
&\leq \big(|x_1| + \|h_r\|_{L^1}\big)|t-\tau| + |p(t) - p(\tau)|,
\end{aligned}$$

and

$$\begin{aligned}
|y_n'(t) - y_n'(\tau)| &\leq \Big|\int_0^t v_n(s)\,ds - \int_0^\tau v_n(s)\,ds\Big| \\
&\leq \Big|\int_\tau^t |v_n(s)|\,ds\Big| \\
&\leq |p'(t) - p'(\tau)|,
\end{aligned}$$

where $p(t) = \int_0^t |\tau - s|h_r(s)ds$ and $p'(t) = \displaystyle\int_0^t h_r(s)\,ds$. From the above inequalities it follows that

$$\begin{aligned}
&\max\{|y_n(t) - y_n(\tau)|, |y_n'(t) - y_n'(\tau)|\} \\
&\leq \max\big\{|x_1| + \|h_r\|_{L^1}\big)|t-\tau| + |p(t)-p(\tau)|, |p'(t) - p'(\tau)|\big\} \\
&\to 0 \quad (n \to \infty),
\end{aligned}$$

which shows that $\{y_n\}$ is an equi-continuous sequence in $Q([a,b])$. Now, $\{y_n\}$ is uniformly bounded and equi-continuous and so it has a cluster point in view of Arzelà-Ascoli theorem. Therefore, the desired conclusion follows by an application of Theorem 2.1. This completes the proof.

To prove the next result, we need the following definitions.

Definition 3.3. A multi-valued function $\beta : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ is called L^1_X-Chandrabhan if

(i) $t \mapsto \beta(t,x,y)$ is measurable for each $x.y \in \mathbb{R}$,
(ii) $\beta(t,x,y)$ is right monotone increasing in x and y for a.e. $t \in J$,
(iii) there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$\|\beta(t,x,y)\|_{\mathcal{P}} = \sup\{|u| : u \in \beta(t,x,y)\} \leq h(t), \text{ a.e. } t \in J$$

for all $x, y \in \mathbb{R}$.

Remark 3.1. Note that, if the multi-valued function $\beta(t,x,y)$ is L^1_X-Chandrabhan, then it is measurable in t and integrally bounded on $J\times\mathbb{R}\times\mathbb{R}$ and so, by a selection theorem, S^1_β has non-empty values, that is,

$$S^1_F(x,y) = \{u \in L^1(J,\mathbb{R}) : u(t) \in F(t,x,y), \text{ a.e. } t \in J\} \neq \emptyset$$

for all $x, y \in \mathbb{R}$ (see Wagner [11] and the references therein).

Theorem 3.2. *Assume that (H_1) and the following hold:*
(H_5) F is L^1_X-Chandrabhan,
Then the DI (1.1) has a solution on J.

Proof. Obviously, the hypotheses (H_2) and (H_3) of Theorem 3.1 hold in view of Remark 3.1. Define two function $a, b : J \to \mathbb{R}$ by

$$a(t) = x_0 + x_1 t - \int_0^t (t-s)h(s)\,ds$$

and

$$b(t) = x_0 + x_1 t + \int_0^t (t-s)h(s)\,ds.$$

It is easy to verify that a and b are, respectively, the lower and upper bounds of the DI (1.1) on J with $a \leq b$. Thus (H_4) holds and now the desired conclusion follows by an application of Theorem 3.1. This completes the proof.

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